Reflecting cones on boolean algebras

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- A poset P is is κ^{op}*-like* if $\forall x \in P \quad |{\uparrow}x| = |\{y \in P : y \geq x\}| < \kappa$.

- A *base* of a space X is a family $\mathscr{B}$ of open sets such that

$$\forall p \in X \quad \forall U \text{ open } \ni p \quad \exists V \in \mathscr{B} \quad p \in V \subseteq U.$$

- For our purposes, all bases are ordered by inclusion. Also, all spaces are Hausdorff.

- The *weight* $w(X)$ of X is

$$\min\{\kappa \geq \omega : \exists \mathscr{B} \text{ base of } X \quad |\mathscr{B}| \leq \kappa\}.$$

  The *order weight* $ow(X)$ of X is

$$\min\{\kappa \geq \omega : \exists \mathscr{B} \text{ base of } X \quad \mathscr{B} \text{ is } \kappa^{\text{op}}\text{-like}\}.$$

Example. Suppose X is a compact metric space. For each $n < \omega$, let $\mathscr{B}_n$ be a finite cover by balls of radius 2^{-n}. Then $\bigcup_{n<\omega} \mathscr{B}_n$ is an ω^{op}-like base of X; hence, $ow(X) = \omega$.

- The *cellularity* $c(X)$ of X is

$$\sup\Big(\{\omega\} \cup \{\kappa : X \text{ has } \kappa\text{-many disjoint open sets}\}\Big).$$

- **van Douwen's Problem.** $c(X) \leq 2^{\aleph_0}$ for all known homogeneous compact X. Is there a counterexample? (After well over twenty years, this is still open in all models of ZFC.)

- Similarly, $ow(X) \leq (2^{\aleph_0})^+$ for all known homogeneous compact X. Is there a counterexample? (After almost one year, this is still open in all models of ZFC.)

- Is there a connection between $c(X)$ and $ow(X)$?

- A compact space X is *dyadic* if it is a continuous image of 2^κ for some κ.

- $c(X) = \omega$ for all compact dyadic X.

- $ow(X)$ can be arbitrarily large for compact dyadic X, but $ow(X) \neq \omega_1$. If X is also homogeneous, then $ow(X) = \omega$.

- In particular, $ow(G) = \omega$ for every compact group G, for every compact group is homogeneous and dyadic (Kuzminov, 1959).

- A *local π-base* at a point p in a space X is a family $\mathscr{B}$ of open sets such that

$$\forall U \text{ open} \ni p \quad \exists V \in \mathscr{B} \quad \emptyset \neq V \subseteq U.$$

  The *π-character* $\pi\chi(p, X)$ of p is

$$\min\{\kappa \geq \omega : \exists \mathscr{B} \text{ local } \pi\text{-base at } p \quad |\mathscr{B}| \leq \kappa\}.$$

- If X is homogeneous, compact, and dyadic, then $\pi\chi(p, X) = w(X)$ for all $p \in X$ (Gerlits, 1976).

- **Theorem 1.** $ow(X) \neq \omega_1$ for all compact dyadic X. Moreover, if $\pi\chi(p, X) = w(X)$ for all $p \in X$, then $ow(X) = \omega$ and every base of X contains an ω^{op}-like base.

Does Theorem 1 hold for any class of nondyadic compact spaces?

- A subset I of a boolean algebra is *independent* if, given any two disjoint finite subsets σ and τ of I, we have $\bigwedge\sigma\wedge\neg\bigvee\tau\neq 0$.

- A boolean algebra is *free* if it is generated by an independent subset.

- A boolean algebra is free iff it is isomorphic to the algebra $\mathrm{Clop}(2^\kappa)$ of clopen subsets of 2^κ for some κ. In particular, $\mathrm{Clop}(2^\kappa)$ is generated by the independent subset

$$\{\{f\in 2^\kappa : f(\alpha)=1\} : \alpha<\kappa\}.$$

- A boolean algebra B *reflects cones* if, for all sufficiently large regular cardinals θ, there is a countable language $\mathcal{L}$ and an $\mathcal{L}$-expansion $\langle H_\theta, \in, \ldots \rangle$ of $\langle H_\theta, \in \rangle$ such that

$$\forall M \prec_{\mathcal{L}} H_\theta \quad \forall p \in B \quad \exists \min(M \cap {\uparrow} p).$$

- Every free boolean algebra reflects cones.

- Denote the Stone dual of a boolean algebra B (*i.e.*, the space of ultrafilters of B) by $\mathrm{st}(B)$.

  Example: $\mathrm{st}(\mathrm{Clop}(2^\kappa)) \cong 2^\kappa$.

- **Theorem 2.** Suppose B reflects cones and X is a continuous image of st(B). Then $ow(X) \neq \omega_1$. Moreover, if $\pi\chi(p, X) = w(X)$ for all $p \in X$, then $ow(X) = \omega$ and every base of X contains an ω^{op}-like base.

- Suppose A and B be boolean algebras. Then st(B) is a a continuous image of st(A) iff B is isomorphic to a subalgebra of A.

- Therefore, Theorem 2 is strictly stronger than Theorem 1 iff there exists a boolean algebra B such that

(∗) B reflects cones but is not a subalgebra of a free boolean algebra.

Is $(*)$ ever satisfied? Not by boolean algebras of size $\leq \aleph_1$. For larger boolean algebras, we have only partial results.

- A boolean algebra B *n-reflects cones* if, for all sufficiently large regular cardinals θ, there is a countable language $\mathcal{L}$ such that given any $p \in B$ and $\in$-chain $M_0, \ldots, M_{n-1}$ satisfying $M_i \prec_{\mathcal{L}} H_\theta$ for all $i < n$, there exists $\min(A \cap \uparrow p)$, where A is a subalgebra of B generated by $B \cap \bigcup_{i<n} M_i$.

- Free boolean algebras n-reflect cones for all $n < \omega$.

- If B n-reflects cones and $|B| \leq \aleph_n$, then B is a subalgebra of a free boolean algebra. If B n-reflects cones for all $n < \omega$, then B is a subalgebra of a free boolean algebra.

- In proving our results, the following lemma, which is based on a technique of Jackson and Mauldin, is heavily used.

- **Lemma.** Let $\mathcal{L}$ be a countable language, β an ordinal, θ a sufficiently large regular cardinal, and $\langle H_\theta, \in, \ldots \rangle$ an $\mathcal{L}$-expansion of $\langle H_\theta, \in \rangle$. Let $\langle M_\alpha \rangle_{\alpha < \beta}$ satisfy

$$|M_\alpha| = \aleph_0 \text{ and } \langle M_\delta \rangle_{\delta < \alpha} \in M_\alpha \prec_{\mathcal{L}} H_\theta$$

for all $\alpha < \beta$. Then, for each $\alpha < \beta$, there is a finite $\in$-chain $N_0, \ldots, N_{k-1}$ such that

$$\bigcup_{i<k} N_i = \bigcup_{\delta<\alpha} M_\delta \text{ and } \forall i < k \quad M_\alpha \ni N_i \prec_{\mathcal{L}} H_\theta.$$

Moreover, if $\beta \leq \omega_{n+1}$, then we can get $k \leq n + 1$.

References

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