Diamond and ultrafilters

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Tukey equivalence

- **Definition/Fact.** A directed set P is Tukey reducible to a directed set Q (written $P \leq_T Q$) if and only if one of the following equivalent statements holds.
  - There is map from P to Q such that the image of every unbounded set is unbounded.
  - There is a map from P to Q such that the preimage of every bounded set is bounded.
  - There is a map from Q to P such that the image of every cofinal subset is cofinal.

- If $P \leq_T Q \leq_T P$, then we say P and Q are Tukey equivalent, writing $P \equiv_T Q$.

- **Theorem** (Tukey, 1940). $P \equiv_T Q$ iff P and Q order-embed as cofinal subsets of a common third directed set.

- Every countable directed set is Tukey-equivalent to 1 (the singleton order) or ω (an ascending sequence).

- The ω_1-sized directed sets are Tukey equivalent to 1, ω, ω_1, $\omega \times \omega_1$ (with the product order), $[\omega_1]^{<\omega}$ (the finite subsets of ω_1 ordered by inclusion), or maybe something else. (E.g., PFA implies these five are exhaustive; CH implies there are 2^{ω_1} more possibilities (Todorčević, 1985).)

What's this got to do with topology?

- **Convention.** Families of open sets are ordered by $\supseteq$.

- **Theorem.** Suppose X and Y are spaces, $p \in X$, $q \in Y$, $\mathcal{A}$ is a local base at p in X, $\mathcal{B}$ is a local base at q in Y, $f\colon X \to Y$ is continuous and open (or just continuous at p and open at p), and $f(p) = q$. Then $\mathcal{B} \leq_T \mathcal{A}$.

- **Proof.** Choose $H\colon \mathcal{A} \to \mathcal{B}$ such that $H(U) \subseteq f[U]$ for all $U \in \mathcal{A}$. (Here we use that f is open.) Suppose $\mathcal{C} \subseteq \mathcal{A}$ is cofinal. For any $U \in \mathcal{B}$, we may choose $V \in \mathcal{A}$ such that $f[V] \subseteq U$ by continuity of f. Then choose $W \in \mathcal{C}$ such that $W \subseteq V$. Hence, $H(W) \subseteq f[W] \subseteq f[V] \subseteq U$. Thus, $H[\mathcal{C}]$ is cofinal.

- **Corollary.** In the above theorem, if f is a homeomorphism, then every local base at p is Tukey-equivalent to every local base at q.

- Thus, the Tukey class of a point's local bases is a topological invariant.

For example, consider the ordered space $X = \omega_1 + 1 + \omega^*$. It has a point p that is the limit of an ascending ω_1-sequence and a descending ω-sequence. Every local base at p (when ordered by by $\supseteq$) is Tukey equivalent to the product order $\omega \times \omega_1$.

Next, consider $D_{\omega_1} \cup \{\infty\}$, the one-point compactification of the ω_1-sized discrete space. Glue X and $D_{\omega_1} \cup \{\infty\}$ together into a new space Y by a quotient map that identifies p and ∞. Think of Y as X with a cloud of points attached to p. In Y, every local base at p is Tukey equivalent to $[\omega_1]^{<\omega}$ (the finite subsets of ω_1 ordered by inclusion), which is not Tukey equivalent to $\omega \times \omega_1$.

Thus, we can distinguish p in X from p in Y by their associated Tukey classes, even though other topological properties, such as character and π-character, have not changed.

The spaces $\beta\omega$ and $\beta\omega \setminus \omega$

- By Stone duality, every ultrafilter $\mathcal{U}$ on ω is such that $\mathcal{U}$ ordered by $\supseteq$ is Tukey-equivalent to every local base of $\mathcal{U}$ in $\beta\omega$.

- Likewise, $\mathcal{U}$ ordered by $\supseteq^*$ (containment mod finite) is Tukey equivalent to every local base of $\mathcal{U}$ in $\beta\omega \setminus \omega$.

- Thus, the classification the Tukey classes of local bases in $\beta\omega$ and $\beta\omega \setminus \omega$ reduces to a problem of infinite combinatorics.

- **Theorem** (Isbell, 1965). There exists $\mathcal{U} \in \beta\omega \setminus \omega$ such that $\langle \mathcal{U}, \supseteq \rangle \equiv_T \langle \mathcal{U}, \supseteq^* \rangle \equiv_T [\mathfrak{c}]^{<\omega}$ (the finite sets of reals ordered by inclusion).

- Every directed set Q of size at most $\mathfrak{c}$ satisfies $1 \leq_T Q \leq_T [\mathfrak{c}]^{<\omega}$, so 1 and $[\mathfrak{c}]^{<\omega}$ are the minimum and maximum Tukey classes among ultrafilters on ω, whether ordered by $\supseteq$ or $\supseteq^*$.

- Every principal ultrafilter is trivially Tukey equivalent to 1.

- **Question** (Isbell, 1965). Is there a $\mathcal{U} \in \beta\omega$ such that

$$1 <_T \langle \mathcal{U}, \supseteq \rangle <_T [\mathfrak{c}]^{<\omega}?$$

Don't take the easy way out.

- For all $\mathcal{U} \in \beta\omega \setminus \omega$, we have $\langle \mathcal{U}, \supseteq^* \rangle \leq_T \langle \mathcal{U}, \supseteq \rangle$. (Proof: use the identity map.)

- If $\mathfrak{u} < \mathfrak{c}$, that is, if some $\mathcal{U} \in \beta\omega \setminus \omega$ has character $\kappa < \mathfrak{c}$, then a trivial cardinality argument shows that

$$1 <_T \langle \mathcal{U}, \supseteq^* \rangle \leq_T \langle \mathcal{U}, \supseteq \rangle \leq_T [\kappa]^{<\omega} <_T [\mathfrak{c}]^{<\omega}.$$

- It's easy to force $\mathfrak{u} < \mathfrak{c}$.

- To make things interesting, we'll restrict our attention to $\mathcal{U} \in \beta\omega \setminus \omega$ with character $\mathfrak{c}$. We'll call the Tukey classes of $\langle \mathcal{U}, \supseteq \rangle$ and $\langle \mathcal{U}, \supseteq^* \rangle$ for such $\mathcal{U}$ "big" Tukey classes.

- Certain Tukey classes just can't occur among local bases in $\beta\omega$ or $\beta\omega \setminus \omega$. Most of the ones below are ruled out by simple cardinality arguments.

  Theorem. Suppose $\mathcal{U} \in \beta\omega \setminus \omega$. Then $\langle \mathcal{U}, \supseteq \rangle$ is not Tukey equivalent to 1, ω, ω_1, $\omega \times \omega_1$, or to any countable union of σ-directed sets. Moreover, $\langle \mathcal{U}, \supseteq^* \rangle$ is not Tukey equivalent to any of 1, ω, $\omega \times \omega_1$, or $\omega \times Q$ where Q is any countable union of σ-directed sets.

- On the other hand, CH implies there exists $\mathcal{U} \in \beta\omega \setminus \omega$ such that $\omega_1 \equiv_T \langle \mathcal{U}, \supseteq^* \rangle <_T \langle \mathcal{U}, \supseteq \rangle$.

- Note that if $\mathcal{U} \in \beta\omega \setminus \omega$, then by definition $\langle \mathcal{U}, \supseteq^* \rangle$ is σ-directed if and only if $\mathcal{U}$ is a P-point in $\beta\omega \setminus \omega$.

- **Main Theorem**. Assuming $\Diamond$, there exists $\mathcal{U} \in \beta\omega \setminus \omega$ such that $\mathcal{U}$ has character $\mathfrak{c}$ and $1 <_T \langle \mathcal{U}, \supseteq^* \rangle \leq_T \langle \mathcal{U}, \supseteq \rangle <_T [\mathfrak{c}]^{<\omega}$. Thus, Isbell's question consistently has a positive answer even when restricted to big Tukey classes.

- $\Diamond$ can be weakened to $\mathsf{MA}_{\sigma\text{-centered}} + \Diamond(S^{\mathfrak{c}}_{\omega})$ where

$$S^{\mathfrak{c}}_{\omega} = \{\alpha < \mathfrak{c} : \operatorname{cf} \alpha = \omega\}.$$

- **Question**. Can $\Diamond$ be weakened to CH? Even a ZFC proof has yet to be ruled out.

About the proof

- For all $\mathcal{U} \in \beta\omega \setminus \omega$, $\langle \mathcal{U}, \supseteq \rangle <_T [\mathfrak{c}]^{<\omega}$ is equivalent to a purely combinatorial statement:

$$\forall \mathcal{A} \in [\mathcal{U}]^{\mathfrak{c}} \quad \exists \mathcal{B} \in [\mathcal{A}]^{\omega} \quad \bigcap \mathcal{B} \in \mathcal{U}.$$

  (For the weaker $\langle \mathcal{U}, \supseteq^* \rangle <_T [\mathfrak{c}]^{<\omega}$, one only needs $\mathcal{B}$ to have a pseudointersection in $\mathcal{U}$.)

- Using $\Diamond$ to diagonalize against all $\mathfrak{c}$-sized subsets of $\mathcal{U}$, we can construct $\mathcal{U} \in \beta\omega \setminus \omega$ such that $\mathcal{U}$ is not a P-point and $\mathcal{U}$ has character $\mathfrak{c}$ and we have that $\forall \mathcal{A} \in [\mathcal{U}]^{\mathfrak{c}}$ $\exists \mathcal{B} \in [\mathcal{A}]^{\omega}$ $\bigcap \mathcal{B} \in \mathcal{U}$.

- Why bother to ensure $\mathcal{U}$ is not a P-point? Because it hasn't been done before. Any P-point $\mathcal{V}$ already satisfies $\langle \mathcal{V}, \supseteq^* \rangle <_T [\mathfrak{c}]^{<\omega}$. To have a non-P-point $\mathcal{U}$ satisfying $\langle \mathcal{U}, \supseteq^* \rangle <_T [\mathfrak{c}]^{<\omega}$ is new.

- More generally, forcing gives us relative freedom in constructing P-points of various Tukey classes. For example, there is a ccc order that forces $\mathfrak{c} = \omega_{42}$ and adds a P-point $\mathcal{V}$ such that $\langle \mathcal{V}, \supseteq^* \rangle \equiv_T \omega_1 \times \omega_{42}$ (Brendle and Shelah, 1999). For non-P-points, equally powerful techniques are yet to be found.

Some questions

- $\Diamond$ implies there are at least three Tukey classes of local bases in $\beta\omega$. Does it imply there are four? infinitely many?

- Is it consistent that there are only two Tukey classes of local bases in $\beta\omega$?

- Is it consistent that there is only one Tukey class of local bases in $\beta\omega \setminus \omega$?

- More ambitiously, is there a model of ZFC with a nice characterization of the Tukey classes of local bases in $\beta\omega$? in $\beta\omega \setminus \omega$?

References

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