Two spectra of Noetherian types

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An order-theoretic weight

Definition

- A **base** of a space X is a set $\mathcal{B}$ of open subsets of X such that for every open subset U of X and every $p \in U$, there exists $B \in \mathcal{B}$ such that $p \in B \subseteq U$.
- The **weight** $w(X)$ of a space X is the least infinite κ such that X has a base of size at most κ.

Definition (Peregudov)

- A family of sets $\mathcal{F}$ is κ^{op}**-like** if every set in $\mathcal{F}$ has fewer than κ-many supersets in $\mathcal{F}$.
- The **Noetherian type** $Nt(X)$ of a space X is the least infinite κ such that X has a κ^{op}-like base.

Easy examples

Theorem

$Nt(X) \leq w(X)^+$ for every space X.

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$Nt(X) \leq w(X)^+$ *for every space* X.

Theorem

$Nt(X) = \omega$ *for every compact metric space* X.

Proof.

For each $n < \omega$, let $\mathcal{U}_n$ be a finite cover of X by balls of radius 2^{-n}. Then $\bigcup_{n<\omega} \mathcal{U}_n$ is an ω^{op}-like base of X. □

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Theorem

$Nt(2^\kappa) = \omega$ for every κ.

Proof.

For each $\sigma \in \mathrm{Fn}(\kappa, 2)$, set $U_\sigma = \{f \in 2^\kappa : \sigma \subseteq f\}$. Then $U_\sigma \supseteq U_\tau$ iff $\sigma \subseteq \tau$. Hence, $\{U_\sigma : \sigma \in \mathrm{Fn}(\kappa, 2)\}$ is an ω^{op}-like base of 2^κ. □

Products

Theorem

If $X = \prod_{i \in I} X_i$ *and each* X_i *has a nontrivial open subset, then* $w(X) = \sum_{i \in I} w(X_i)$.

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Theorem (Malykhin)

If $X = \prod_{i\in I} X_i$, $w(X) \leq |I|$, *and each* X_i *is a nontrivial union of two open sets, then* $Nt(X) = \omega$.

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If $X = \prod_{i\in I} X_i$*,* $w(X) \leq |I|$*, and each* X_i *is a nontrivial union of two open sets, then* $Nt(X) = \omega$.

Question

Are there a spaces X, Y such that $Nt(X \times Y) < Nt(X)Nt(Y)$?

$\beta\mathbb{N} \setminus \mathbb{N}$

Theorem (M.)

Each of the following is consistent with ZFC.

- $Nt(\beta\mathbb{N} \setminus \mathbb{N}) = \omega_1 = 2^\omega$ *(Malykhin)*
- $\omega_1 < Nt(\beta\mathbb{N} \setminus \mathbb{N}) = 2^\omega$
- $Nt(\beta\mathbb{N} \setminus \mathbb{N}) = \omega_1 < 2^\omega$
- $\omega_1 < Nt(\beta\mathbb{N} \setminus \mathbb{N}) < 2^\omega$
- $\omega_1 < 2^\omega < (2^\omega)^+ = Nt(\beta\mathbb{N} \setminus \mathbb{N})$

Much more can be said in terms of $\mathfrak{s}$, $\mathfrak{r}$, $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{u}$, $\mathfrak{h}$, $\mathfrak{i}$, and $\mathfrak{p}$. (*E.g.*, $Nt(\beta\mathbb{N} \setminus \mathbb{N}) \geq \mathfrak{s}$ and $\mathrm{Con}(\mathfrak{b} < Nt(\beta\mathbb{N} \setminus \mathbb{N}) < \mathfrak{d})$.)

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Theorem (M.)

Suppose $|I| < 2^\omega$ and $w(X_i) \leq 2^\omega$ for all $i \in I$. Then $\prod_{i \in I}(X_i \oplus (\beta\mathbb{N} \setminus \mathbb{N}))$ is not homeomorphic to a product of 2^ω-many non-singleton spaces.

Van Douwen's Problem

Definition

- A **compactum** is a compact Hausdorff space.
- A space is **homogeneous** is for every $p, q \in X$ there is an homeomorphism $h\colon X \to X$ such that $h(p) = q$.
- The **cellularity** $c(X)$ of a space X is the least infinite κ such that every pairwise disjoint open family is X has size at most κ.

Question (Van Douwen)

Is there a homogeneous compactum X such that $c(X) > 2^\omega$?

After over 30 years, Van Douwen's Problem has not been solved in any model of ZFC.

The difficulty is structural

Fact

Every known example of a homogeneous compactum X is a continuous image of a product of compacta each with weight at most 2^ω. Hence, $c(X) \leq 2^\omega$. (The upper bound is attained.)

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Theorem A (M.)

If X is a homogeneous compactum and a continuous image of a product $\prod_{i \in I} X_i$ of compacta such that

$$\sup_{i \in I} w(X_i) = \kappa < \operatorname{cf} \lambda = \lambda \leq w(X),$$

then $Nt(X) \leq \kappa$.

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If X is a homogeneous compactum and a continuous image of a product $\prod_{i \in I} X_i$ of compacta such that

$$\sup_{i \in I} w(X_i) = \kappa < \operatorname{cf} \lambda = \lambda \leq w(X),$$

then $Nt(X) \leq \kappa$.

Corollary

Every known homogeneous compactum X satisfies $Nt(X) \leq (2^\omega)^+$.

(The upper bound is attained.)

Dyadic compacta

Definition

A **dyadic compactum** is a Hausdorff space that is a continuous image of 2^κ for some κ.

Corollary

$Nt(X) = \omega$ for every homogeneous dyadic compactum X.

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Theorem (M.)

Let $\kappa < \lambda$ be infinite cardinals and let X be the quotient of $2^\kappa \oplus 2^\lambda$ obtained by identifying $\langle 0 \rangle_{i<\kappa}$ and $\langle 0 \rangle_{i<\lambda}$. If $\kappa < \operatorname{cf} \lambda$, then $Nt(X) = \lambda^+$. If $\kappa = \operatorname{cf} \lambda$, then $Nt(X) = \lambda$.

Corollary

The class of Noetherian types of dyadic compacta includes all infinite cardinals except possibly weak inaccessibles and successors of cardinals with countable cofinality (like ω_1 and $\omega_{\omega+1}$).

Non-triviality

Theorem (M.)

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Theorem B (M.)

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Theorem B (M.)

The class of Noetherian types of dyadic compacta excludes ω_1.

Question

Are $\omega_{\omega+1}$ and weak inaccessibles excluded too?

Another spectrum

Perhaps things are easier with ordered compacta.

Theorem (M.)

With respect to the order topology, $Nt(\kappa+1)=\kappa^+$ if κ is a regular uncountable cardinal and $Nt(\kappa+1)=\kappa$ if κ is a singular cardinal.

Proof.

By the Pressing Down Lemma… $\square$

Excluding ω_1

Theorem C (M.)

If X is an ordered compactum and $Nt(X) \leq \kappa$ and κ is regular and uncountable, then X has a dense set of size less than κ.

Corollary

If X is an ordered compactum, then $Nt(X) \neq \omega_1$.

Excluding ω_1

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If X is an ordered compactum, then $Nt(X) \neq \omega_1$.

Proof.

Suppose $Nt(X) = \omega_1$. Then X has a countable dense subset D. Also, X is not metrizable, so $w(X) \geq \omega_1$. Let $\mathcal{B}$ be a base of X. Then for some $p, q \in D$ and $U \in \mathcal{B}$ we have $U \subseteq (p, q) \subseteq V$ for uncountably many $V \in \mathcal{B}$. Thus, $Nt(X) \geq \omega_2$. □

Excluding weak inaccessibles

Corollary

If X is an ordered compactum, then $Nt(X)$ is not a weak inaccessible.

Proof.

Suppose $\kappa = Nt(X)$ is weakly inaccessible. Then X has a dense subsets D of size less than κ. If $w(X) \geq \kappa$, then, arguing as before, $Nt(X) \geq \kappa^+$. If $w(X) < \kappa$, then $Nt(X) \leq w(X)^+ < \kappa$. □

A spectrum characterized

Theorem D (M.)

For each singular cardinal κ, there is an ordered compactum X such that $Nt(X) = \kappa^+$.

Corollary

The class of Noetherian types of ordered compacta includes all infinite cardinals except ω_1 *and the weak inaccessibles.*

Questions

Question

Do the dyadic compacta have the same Noetherian spectrum as the ordered compacta?

Question

Is there an interesting example of a class of spaces with Noetherian spectrum excluding ω_2?

Proving Theorems A and B

Theorem A

If X is a homogeneous compactum and a continuous image of a product $\prod_{i \in I} X_i$ of compacta such that

$$\sup_{i \in I} w(X_i) = \kappa < \operatorname{cf} \lambda = \lambda \leq w(X),$$

then $Nt(X) \leq \kappa$.

Theorem B

The class of Noetherian types of dyadic compacta excludes ω_1.

Substructures and quotients

Definition

- $H(\theta)$ is the set of sets that are hereditarily smaller than θ.
- $C(X)$ is the set of continuous maps from X to $\mathbb{R}$.
- Given X a compactum, θ a sufficiently large regular cardinal, and M an elementary substructure of $\langle H(\theta), \in, <, C(X) \rangle$, define a quotient map $\pi^X_M \colon X \to X/M$ by

$$\pi^X_M(p) \neq \pi^X_M(q) \text{ iff } f(p) \neq f(q) \text{ for some } f \in C(X) \cap M.$$

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- Given a base $\mathcal{B}_\alpha$ of each X/M_α, $\mathcal{B} = \bigcup_{\alpha < w(X)} \left(\pi^X_{M_\alpha}\right)^{-1}(\mathcal{B}_\alpha)$ is a base of X.

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- Given a base $\mathcal{B}_\alpha$ of each X/M_α, $\mathcal{B} = \bigcup_{\alpha < w(X)} \left(\pi^X_{M_\alpha}\right)^{-1}(\mathcal{B}_\alpha)$ is a base of X.
- If the substructures cohere sufficiently well, then we can use reflection arguments (and $\min_{p \in X} \pi\chi(p, X) = w(X)$) to carefully construct subsets $\mathcal{A}_\alpha \subseteq \mathcal{B}_\alpha$ such that $\mathcal{A} = \bigcup_{\alpha < w(X)} \left(\pi^X_{M_\alpha}\right)^{-1}(\mathcal{A}_\alpha)$ is an ω^{op}-like base of X.

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It suffices to have $\langle M_\beta \rangle_{\beta < \alpha} \in M_\alpha$ for all $\alpha < w(X)$.

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Let Ω denote the class of finite, nonempty ordinal sequences $\langle \gamma_i \rangle_{i<n}$ for which $|\gamma_0| > |\gamma_1| > \cdots > |\gamma_{n-2}| \geq \omega_1 > |\gamma_{n-1}|$ if $n \geq 2$ and $\omega_1 > |\gamma_{n-1}|$ if $n = 1$.

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Let $\sqsubseteq$ denote the lexicographic ordering of Ω. Since $\sqsubseteq$ is a well-ordering, there is a unique isomorphism Υ from the ordinals to $\langle \Omega, \sqsubseteq \rangle$. For each $\Upsilon(\alpha) = \langle \gamma_i \rangle_{i<n}$ and $k < n-1$, define:

$$N_{\alpha,k} = \bigcup \{ M_\beta : \langle \gamma_0, \ldots, \gamma_{k-1}, 0 \rangle \sqsubseteq \Upsilon(\beta) \sqsubset \langle \gamma_0, \ldots, \gamma_k, 0 \rangle \}$$

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Coherence as promised

Theorem

- $\bigcup_{i<n} N_{\alpha,i} = \bigcup_{\beta<\alpha} M_\alpha$
- $N_{\alpha,i} \in M_\alpha$ *for all* $i < n$.
- $N_{\alpha,i} \prec H(\theta)$ *for all* $i < n$.

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Jackson and Mauldin first constructed a tree of substructures satisfying the above theorem. I just showed that one can build the tree from a mere sequence of M_α's satisfying $\langle M_\beta \rangle_{\beta<\alpha} \in M_\alpha$.

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Remark

If $w(X) = \omega_1$, then we don't need such fancy machinery; a continuous elementary chain of countable submodels suffices.

Proving Theorem C

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Proof.

- Let $\mathcal{B}$ be κ^{op}-like base of X. Let $M \prec \langle H(\theta), \in, \mathcal{B} \rangle$, $|M| < \kappa$, and $M \cap [H(\theta)]^{<\kappa} \subseteq [M]^{<\kappa}$.

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- If $p, q \in X \cap M$ and $\emptyset \neq (p, q) \subseteq U \in \mathcal{B}$, then $U \in M$, so points like $\min\{x \in X : q \leq x \notin U\}$ are also in M.

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- If $p, q \in X \cap M$ and $\emptyset \neq (p, q) \subseteq U \in \mathcal{B}$, then $U \in M$, so points like $\min\{x \in X : q \leq x \notin U\}$ are also in M.
- It follows that $X \cap M$ is dense in X.

□

Proving Theorem D

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- Let $\lambda = \operatorname{cf} \kappa$ and $Y = \lambda^+ + 1$.

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- Partition the limit ordinals less than λ^+ into stationary sets $\langle S_\alpha \rangle_{\alpha<\lambda}$.
- Let $\langle \kappa_\alpha \rangle_{\alpha<\lambda}$ be an increasing sequence of regular cardinals with supremum κ.

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- For each $\alpha < \lambda$ and $\beta \in S_\alpha$, set $Z_\beta = (\kappa_\alpha + 1)^{\mathrm{op}}$.

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- For each $\alpha < \lambda$ and $\beta \in S_\alpha$, set $Z_\beta = (\kappa_\alpha + 1)^{\mathrm{op}}$.
- For each $\beta \in Y \setminus \bigcup_{\alpha<\lambda} S_\alpha$, set $Z_\beta = 1$.

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- For each $\alpha < \lambda$ and $\beta \in S_\alpha$, set $Z_\beta = (\kappa_\alpha + 1)^{\mathrm{op}}$.
- For each $\beta \in Y \setminus \bigcup_{\alpha < \lambda} S_\alpha$, set $Z_\beta = 1$.
- $X = \sum_{\beta \in Y} Z_\beta$.

□