Order types of bases in box products, ω^*, and homogeneous compacta

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Convention

All spaces are T_3 (regular and Hausdorff).

A motivating example

A space X is **homogeneous** if for all points p, q there is a homeomorphism $h\colon X \mapsto X$ sending p to q.

(Maurice, 1964)

- Let $X = 2^{\omega^2}_{\mathrm{lex}}$ be the binary sequences of ordinal length ω^2 ordered lexicographically.
- X is compact and homogeneous.
- X has a big family of pairwise disjoint open sets:
  - For each $g \in 2^\omega$, let $U_g = \{f \in X : g000\ldots < f < g111\ldots\}$.
- More precisely, X has a **cellular family** of size $2^{\aleph_0}$.

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- More precisely, X has a **cellular family** of size $2^{\aleph_0}$.

- We can replace ω^2 with ω^α for any countable ordinal α.
- We cannot go further: compact homogeneous linear orders cannot have increasing (or decreasing) uncountable sequences.

Van Douwen's Problem: open for over forty years

Let the **cellularity** of X, or $c(X)$, be the supremum of the cardinalities of its pairwise disjoint open families.

Is there a compact homogeneous space X with $c(X) > 2^{\aleph_0}$?

We don't know the answer in any model of set theory.

If we try adding structure to enforce homogeneity...

Adding first-order structure hasn't solved Van Douwen's Problem.

For example, every compact group has a (left or right) Haar probability measure, and therefore has countable cellularity.

(Hart-Kunen, 1999) If we replace "group" with "quasigroup" or various other first-order structures that enforce homogeneity, then the resulting compacta still have countable cellularity.

If we try transfinite brute force. . .

Can we iteratively modify a space, adding autohomeomorphisms until we're done?

For first countable zero-dimensional spaces, homogeneity of the space is equivalent to homogeneity of the clopen algebra (a first-order structure).

Only in this setting has transfinite brute force built compact homogeneous spaces.

(Arhangel′skiĭ's Theorem) First countable compact spaces cannot have more than $2^{\aleph_0}$ points.

If we try large products...

Homogeneous factors

- The cellularity of a product is the supremum of the cellularity of its finite subproducts.
- (M., 2006) All known examples of homogeneous compacta (mostly compact groups and first countable homogeneous compacta) are continuous images of products whose factors all have bases of size $\leq 2^{\aleph_0}$.
- Therefore, products of known homogeneous compacta cannot have cellularity $> 2^{\aleph_0}$.

If we try large products. . .

Inhomogeneous factors

- (Dow-Pearl, 1997) Infinite powers of first countable zero-dimensional spaces are homogeneous.
- The hypothesis of first countability cannot be removed, *e.g.*, no power of $\omega_1 + 1$ is homogeneous.
- (Many authors) Many theorems about the class of homogeneous spaces (*e.g.*, $|X| \leq 2^{\pi\chi(X)c(X)}$) have been proven true of the power homogeneous spaces (*i.e.*, those spaces X for which some X^κ is homogeneous).
- (Kunen, 1990) Products of infinite compact F-spaces (*e.g.*, $\beta\omega \setminus \omega$) are not homogeneous.
- (Arhangel′skiĭ, 2005) A product of compact linear orders is not homogeneous unless all factors are first countable.

Enter order theory

Rectangular local bases imply large cellularity.

- **Convention:** Subsets of spaces are ordered by $\supseteq$.
- If a point in a space has a local base $\mathcal{B}$ or order type $\omega \times \omega_1 \times \omega_2$, then that space has a cellular family of size $\aleph_2$ (and CH implies $\aleph_2 > 2^{\aleph_0}$).
- All points in $X = 2^{\omega}_{\mathrm{lex}} \times 2^{\omega_1}_{\mathrm{lex}} \times 2^{\omega_2}_{\mathrm{lex}}$ have local bases of order type $\omega \times \omega_1 \times \omega_2$.
- X is compact but not, alas, homogeneous. (Some, but not all, points have countable local π-bases.)

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- All points in $X = 2^{\omega}_{\mathrm{lex}} \times 2^{\omega_1}_{\mathrm{lex}} \times 2^{\omega_2}_{\mathrm{lex}}$ have local bases of order type $\omega \times \omega_1 \times \omega_2$.
- X is compact but not, alas, homogeneous. (Some, but not all, points have countable local π-bases.)
- However, we haven't proved that no homogeneous compacta can have a local base with order type $\omega \times \omega_1 \times \omega_2$.
- Also, proving that would be very interesting in itself.

Cofinal types vs. order types.

Two preorders P, Q are **cofinally equivalent** (written $P \equiv_{\mathrm{cf}} Q$) if there is a preorder R with cofinal subsets P', Q' order-isomorphic to P, Q (respectively).

E.g., $\mathbb{Q} \equiv_{\mathrm{cf}} \{\sqrt{n} : n \in \mathbb{N}\}$ because both are cofinal in $\mathbb{R}$.

Less trivially, $P = \omega \times \omega_1$ (with the product order $x \leq y \Leftrightarrow x_0 \leq y_0 \wedge x_1 \leq y_1$) is cofinally equivalent to $Q = (\omega \times \omega_1, \trianglelefteq)$ where $x \triangleleft y \Leftrightarrow x_0 < y_0 \wedge x_1 < y_1$, even though P has uncountable chains and no infinite antichains, while Q has uncountable antichains and no uncountable chains.

(For directed sets (*e.g.*, local bases), cofinal equivalence $\Leftrightarrow$ Tukey equivalence.)

Advantages of cofinal types

In general, two local bases at the same point may have different order types, but they always have the same cofinal type.

Instead of considering order types of particular local bases at $p \in X$, we only consider the cofinal type of $\mathrm{Nbhd}(p, X)$, the set of all neighborhoods of p in X.

The cofinal type of $\mathrm{Nbhd}(p, X)$ does not change if switch the ordering from $\supseteq$ to $\sqsupseteq$ where $U \sqsupset V$ means the interior of U contains the closure of V.

Rectangular cofinal types and cellularity

If $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_1 \times \omega_2$, then $c(X) \geq \aleph_2$.

More generally:

- Definition: $\mathrm{cf}(P)$ is the least of the sizes of cofinal subsets of P.
- If $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} P \times Q$, $\mathrm{cf}(P) < \kappa$, and all subsets of Q of size $< \kappa$ are bounded (above), then X has a cellular family of size κ.

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Questions

- Does any point p in any infinite homogeneous compactum X satisfy...
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  - $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega_1 \times \omega_2$? No, because every countably infinite $A \subseteq X$ has a limit point in X.

Skinny rectangles

Theorem (M., 2011)

Assume CH or PFA. If X is compact, then not all $p \in X$ satisfy $\mathrm{Nbhd}(p, X) \equiv_{\mathrm{cf}} \omega \times \omega_2$.

Question

- Can the "CH or PFA" hypothesis be dropped?
- Probably. ZFC proves that not all $p \in X$ can have a clopen local base of order type $\omega \times \omega_2$.

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A ZFC theorem that is weaker at $\omega \times \omega_2$, but more general:

If X is compact and κ is an infinite regular cardinal, then not all $y \in X$ satisfy

- $\mathrm{Nbhd}(y, X) \equiv_{\mathrm{cf}} P_y \times Q_y$,
- $\mathrm{cf}(P_y) < \kappa$, and
- all subsets of Q_y of size $\leq 2^{<\kappa}$ are bounded.

Skinny rectangles in $\omega^* = \beta\omega \setminus \omega$

Not all neighborhood filters are rectangles.

Not all $\mathrm{Nbhd}(p, \omega^*)$ are $\equiv_{\mathrm{cf}}$ some $\kappa_0 \times \cdots \times \kappa_n$.

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Can there be rectangles? How skinny can they be?

- $\aleph_1 \leq \mathfrak{m}_{\sigma-2-\mathrm{linked}} \leq \mathfrak{m}_{\sigma-3-\mathrm{linked}} \leq \ldots \leq \mathfrak{m}_{\sigma-\mathrm{centered}} \leq 2^{\aleph_0}$.
- $\mathrm{MA} \Rightarrow \mathrm{MA}_{\sigma-2-\mathrm{linked}} \Leftrightarrow \mathfrak{m}_{\sigma-2-\mathrm{linked}} = 2^{\aleph_0}$.
- It is consistent that $\sup_{2 \leq n < \omega} \mathfrak{m}_{\sigma-n-\mathrm{linked}} < \mathfrak{m}_{\sigma-\mathrm{centered}}$.

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- (M., 2007) For all regular uncountable κ, it is consistent with ZFC that $\mathfrak{m}_{\sigma-\mathrm{centered}} = \kappa \ll 2^{\aleph_0}$ and some $p \in \omega^*$ satisfies $\mathrm{Nbhd}(p, \omega^*) \equiv_{\mathrm{cf}} \kappa \times 2^{\aleph_0}$.

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- What happens between $\sup_{2 \leq n < \omega} \mathfrak{m}_{\sigma-n-\mathrm{linked}}$ and $\mathfrak{m}_{\sigma-\mathrm{centered}}$?

Skinny rectangles as gaps in a spectrum

$\omega \times \omega_2$ has unbounded increasing sequences of lengths ω and ω_2, but not of length ω_1. Can such a gap appear in every $\text{Nbhd}(p, X)$?

Theorem (M., 2011)

- Let X be compact and λ be an infinite regular cardinal.
- Let $\text{Escape}(p, X)$ be the set of regular cardinals κ for which there exists a κ-long sequence $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\alpha \supseteq \cdots$ in $\text{Nbhd}(p, X)$ such that p is not in the interior of $\bigcap_{\alpha<\kappa} U_\alpha$.
- Suppose that $\lambda \notin \bigcup_{p \in X} \text{Escape}(p, X)$.
- It follows that X has no λ-long free sequence; hence, some $q \in X$ has a local base of size at most 2^μ, for some $\mu < \lambda$.

Corollary

Assume GCH. For all infinite compact homogeneous X and all $p \in X$, $\text{Escape}(p, X)$ is an initial segment of the class of infinite regular cardinals and closed at inaccessibles.

Strongly unbounded sets

- A subset S of a preorder P is **strongly unbounded** if S is infinite and every infinite subset of S is unbounded (from above).
- Finite products of ordinals (*e.g.*, $\omega \times \omega_1$) never have uncountable strongly unbounded subsets.

(M., 2006) Applications to compact homogeneous X:

- If some local base (ordered by $\supseteq$) lacks a strongly unbounded set of size $\log|X|$, then $c(X) \geq \log|X|$.
- In every known example of X, every local base has a cofinal strongly unbounded subset (of size $\chi(X) \geq \log|X|$).

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- In every known example of X, every local base has a cofinal strongly unbounded subset (of size $\chi(X) \geq \log |X|$).
- $\pi\chi(X) \geq \kappa \Rightarrow$ every local base has a strongly unbounded subset of size κ.
- $\pi\chi(X) = \chi(X) \Rightarrow$ every local base has a cofinal strongly unbounded subset.

κ-strongly unbounded bases

- A preorder P is κ-**strongly unbounded** (in itself) if P has size $\geq \kappa$ and every subset of P of size κ is unbounded.
- (M., 2006) Every known homogeneous compactum has a $\left(2^{\aleph_0}\right)^+$-strongly unbounded base or a base of size $\leq 2^{\aleph_0}$.

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An incomplete analogy

- X_δ is X with all G_δ-sets declared open.
- (Juhász, 1972) If X is compact, then $c\left(X_\delta\right) \leq 2^{c(X)}$.
- (Spadaro, 2009) Assume GCH. If X is compact, $\mathrm{cf}(\kappa)$ is uncountable, and X has a κ-strongly unbounded base, then X_δ has a 2^κ-strongly unbounded base. What if $\kappa = \aleph_0$?

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- Every infinite power of 2 has an $\aleph_0$-strongly unbounded base.
- $2_\delta^{\aleph_n}$ has an $\aleph_1$-strongly unbounded base, for all $n < \omega$.
- Does $2_\delta^{\aleph_\omega}$ have an $\aleph_1$-strongly unbounded base?

Enter large cardinals

Theorem (M., 2009)

- $2_\delta^{\aleph_\omega}$ has a $\left(2^{\aleph_0}\right)^+$-strongly unbounded base.
- It is consistent, relative to ZFC, that $2_\delta^{\aleph_\omega}$ has an $\aleph_1$-strongly unbounded base. (We can directly add the base by forcing.)

Theorem (Levinski-Magidor-Shelah, 1990)

$CC_{\aleph_\omega}$, the Chang conjecture variant $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$, is consistent with ZFC+GCH, relative to a 2-huge cardinal.

Theorem (Soukup, 2009)

If $CC_{\aleph_\omega}$, then $2_\delta^{\aleph_\omega}$ does not have an $\aleph_1$-strongly unbounded local base.

Corollary

$2_\delta^{\aleph_\omega}$ can consistently lack a $2^{\aleph_0}$-strongly unbounded base.

Enter the approachability ideal $I[\lambda]$

Theorem (Kojman-M.-Spadaro, 2010)

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Theorem (Kojman-M.-Spadaro, 2010)

1. $2^{\aleph_\omega}_\delta$ has an $\aleph_4$-strongly unbounded base.
2. MM implies that $2^{\aleph_\omega}_\delta$ has an $\aleph_2$-strongly unbounded base.
3. $\square_{\aleph_\omega}$ and $\aleph_{\omega+1} = \mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq)$ together imply that $2^{\aleph_\omega}_\delta$ has an $\aleph_1$-strongly unbounded base.

Regarding the proof: For (1), we use the main idea of Shelah's proof that there is a stationary set in $I[\lambda] \restriction S^\lambda_\kappa$ for all regular λ and κ with $\lambda \geq \kappa^{+3}$. (2) and (3) are instances of $2^{\aleph_\omega}_\delta$ having a κ-strongly unbounded base if $S_\kappa^{\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq)} \in I\left[\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq)\right]$.

Questions

- Does MM imply that $2^{\aleph_\omega}_\delta$ lacks an $\aleph_1$-strongly unbounded base?
- Is it consistent with ZFC that $2^{\aleph_\omega}_\delta$ lacks an $\aleph_2$-strongly unbounded base?

π-bases

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We know $CC_{\aleph_\omega}$ prohibits $\aleph_1$-strongly unbounded local bases of $2_\delta^{\aleph_\omega}$, but does it prohibit $\aleph_1$-strongly unbounded π-bases? (Assume the consistency of a 2-huge cardinal.)

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Theorem (M.-Spadaro, 2011)

$\mathsf{CC}_{\aleph_\omega}$ and $\aleph_\omega < 2^{\aleph_0}$ are mutually consistent; $\aleph_\omega < 2^{\aleph_0}$ implies that $2^{\aleph_\omega}_\delta$ has an $\aleph_1$-strongly unbounded π-base.

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Question

What happens in models of $CC_{\aleph_\omega}$ and $2^{\aleph_0} < \aleph_\omega$?