

# Higher-arity properties of inverse limit systems

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## Basic terminology and scope

**compactum** = compact Hausdorff space

**Boolean space** = compactum with a clopen base

**dyadic** = continuous image of a power of 2

**crowded** = without isolated points

**ccc** = every pairwise disjoint family of open sets is countable.

- ▶ Most of our example spaces will be boolean and dyadic.
- ▶ All of our example spaces will be crowded ccc compacta.

**open map** = function that maps open sets onto open sets

## Some notation

$X \cong Y$  means  $X$  is homeomorphic to  $Y$ .

$\pi_0$  and  $\pi_1$  are the first and second coordinate projections:

$$\pi_0(x, y) = x \text{ and } \pi_1(x, y) = y$$

$h = f \times g$  means  $h$  is the **diagonal product** of  $f$  and  $g$ :

$$h(x) = (f(x), g(x))$$

$Z = f \boxtimes g$  means  $Z$  is the **fiber product** of  $f$  and  $g$ :

$$Z = \{(x, y) : f(x) = g(y)\}$$

## Prelude: symmetric powers of $2^\kappa$

- ▶  $2^\kappa$  is space of all functions from  $\kappa$  to 2.
- ▶  $(2^\kappa)^n$  is space of all functions from  $n$  to  $2^\kappa$ .
- ▶  $SP^n(2^\kappa)$  is the quotient space of  $(2^\kappa)^n$  induced by:  
$$f \sim g \text{ iff } f = g \circ \tau \text{ for some permutation } \tau: n \rightarrow n$$
- ▶ Given  $0 < m < n < \omega \leq \kappa$ , we have  $(2^\kappa)^m \cong (2^\kappa)^n$  since  $|\kappa \times m| = |\kappa \times n|$ .
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- ▶ (Ščepin) If  $\kappa \geq \omega_2$  then  $SP^m(2^\kappa) \not\cong SP^n(2^\kappa)$ . In fact, Ščepin's proof shows that  $SP^n(2^\kappa)$  is not a retract of any homeomorphic copy of  $SP^m(2^\kappa)$ .



So, why isn't, say,  $SP^7(2^{\omega_2})$  a retract of  $SP^6(2^{\omega_2})$ ?

### Lemma

Given a cardinal  $\kappa$ , the following are equivalent:

- ▶  $\kappa \geq \omega_2$ .
- ▶ There exist  $M, N \prec H(\theta)$  such that  $M \cap \kappa \not\subseteq N \cap \kappa \not\subseteq M \cap \kappa$ .
- ▶ There exist **countable**  $M, N \prec H(\theta)$  such that  $M \cap \kappa \not\subseteq N \cap \kappa \not\subseteq M \cap \kappa$ .

### Definition

“ $M \prec H(\theta)$ ” implies that  $M$  satisfies arbitrary finite lists of finitary closure properties that I don't want to write down. More precisely:

- ▶  $M \prec H(\theta)$  means  $(M, \in \upharpoonright M, \sqsubset \upharpoonright M)$  is a first-order **elementary submodel** of a set-theoretic universe well-ordered by  $\sqsubset$ .
- ▶ To avoid going beyond ZFC, choose a “universe” of the form  $(H(\theta), \in, \sqsubset)$  for  $\theta$  a regular cardinal large enough that  $H(\theta)$  has all the power sets we need for the argument at hand.

## An impossible fiber retract

Let  $SP^7(2^{\omega_2}) \xrightarrow{f} SP^6(2^{\omega_2}) \xrightarrow{g} SP^7(2^{\omega_2})$ .  
 $\text{id}$

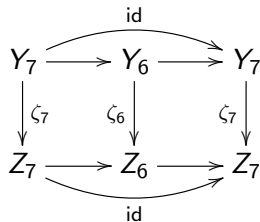
Let  $N_0, N_1 \prec H(\theta)$  with  $f, g \in N_i$  and  $E_i = N_i \cap \omega_2 \not\subseteq E_{1-i}$ .

Let  $Y_j = SP^j(2^{E_0 \cup E_1})$  for each  $j \in \{6, 7\}$ .

Let  $Z_j$  be the fiber product  $\pi_{j, E_0 \cap E_1}^{E_0} \boxtimes \pi_{j, E_0 \cap E_1}^{E_1}$

where  $\pi_{j, A}^B: SP^j(2^B) \rightarrow SP^j(2^A)$ ,  $f/\sim \mapsto ((f_i \upharpoonright A)_{i < j})/\sim$ .

Then the diagonal product  $\zeta_j = \pi_{j, E_0}^{E_0 \cup E_1} \times \pi_{j, E_1}^{E_0 \cup E_1}$  maps  $Y_j$  into  $Z_j$ .



- ▶  $\zeta_7$  has a fiber of size 7!
- ▶  $\zeta_6$  has no fibers larger than 6!
- ▶ Contradiction!

## Why are there fibers of size 7!?

Let  $a \in (2^{E_0 \cap E_1})^7$  and  $a_i = a_j$  for all  $i < j < 7$ .

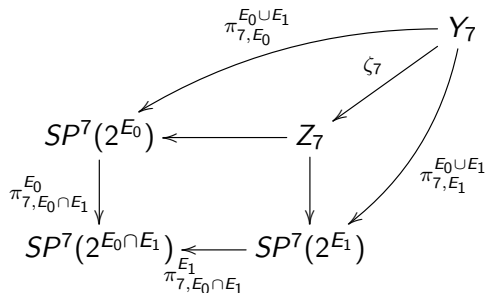
Let  $b \in (2^{E_0 \setminus E_1})^7$  and  $b_i \neq b_j$  for all  $i < j < 7$ .

Let  $c \in (2^{E_1 \setminus E_0})^7$  and  $c_i \neq c_j$  for all  $i < j < 7$ .

Then, for all permutations  $\rho, \sigma: 7 \rightarrow 7$ :

$$\begin{aligned} \zeta_7((a_i \cup b_{\rho(i)} \cup c_{\sigma(i)})_{i < 7}) / \sim &= \zeta_7((a_i \cup b_i \cup c_{\tau(i)})_{i < 7}) / \sim \\ &= ((a_i \cup b_i)_{i < 7} / \sim, (a_i \cup c_{\tau(i)})_{i < 7} / \sim) \\ &= ((a_i \cup b_i)_{i < 7} / \sim, (a_i \cup c_i)_{i < 7} / \sim) \end{aligned}$$

where  $\tau = \sigma \circ \rho^{-1}$ .



## What new symmetries break at $\omega_3$ ?

“Symmetry breaking” at  $\omega_2$  is not just for symmetric powers. For example,  $2^{\omega_1}$  is homeomorphic to its Vietoris hyperspace  $\exp(2^{\omega_1})$  (Ščepin) but  $\exp(2^{\omega_2})$  is not a continuous image of  $2^{\omega_2}$  (Shapiro).

Are there analogous phenomena at  $\omega_3$ ?

Bell’s non-supercompact dyadic compactum of weight  $\omega_3$  is loosely analogous. However, we don’t know if  $\omega_3$  is least possible.

At least our lemma about  $\omega_2$  has a clear analog:

### Lemma

*Let  $0 < d < \omega$  and let  $\kappa$  be a cardinal. The following are equivalent.*

- ▶  $\kappa \geq \omega_d$ .
- ▶ There exist  $N_0, \dots, N_{d-1} \prec H(\theta)$  such that  $\kappa \cap \bigcap_{j \neq i} N_j \not\subseteq N_i$  for all  $i < d$ .
- ▶ There exist **countable**  $N_0, \dots, N_{d-1} \prec H(\theta)$  such that  $\kappa \cap \bigcap_{j \neq i} N_j \not\subseteq N_i$  for all  $i < d$ .

## Quotients induced by elementary submodels

Suppose that  $X$  is  $T_{3.5}$ ,  $I$  is a set, and  $X \in N_i \prec H(\theta)$  for all  $i \in I$ .  
(By “ $X \in N_i$ ” we actually mean “the topology of  $X$  is an element of  $N_i$ .”)

### Definition

$p \not\sim q$  if  $f(p) \neq f(q)$  for some  $f \in C(X, \mathbb{R}) \cap \bigcup_{i \in I} N_i$ .

Let  $X / \bigcup_{i \in I} N_i$  be the quotient space induced by  $\sim$ .

Let  $Q_{\bigcup_{i \in I} N_i}$  be the associated quotient map.

### Lemma

*If  $X$  is a compactum, or even just  $T_4$ , then the following are equivalent:*

- ▶  $p \not\sim q$
- ▶  $p$  and  $q$  have disjoint closed neighborhoods  $U, V \in N_i$  for some  $i$ .
- ▶  $p$  and  $q$  have disjoint closed  $G_\delta$  neighborhoods  $U \in N_i$  and  $V \in N_j$  for some  $i, j$ .

## Open quotient maps

When is the quotient map  $Q_S$  an open map?

### Example

Given  $X = \omega_1 + 1$  and a countable  $N \prec H(\theta)$ , we have:

- ▶ There is a natural homeomorphism  $h: X/N \cong \delta + 1$  where  $\delta = \omega_1 \cap N$ :

$$h(Q_N(\alpha)) = \begin{cases} \alpha & \text{if } \alpha < \delta \\ \delta & \text{if } \alpha \geq \delta \end{cases}$$

- ▶  $h \circ Q_N$  sends the isolated point  $\delta + 1$  to the limit point  $\delta$ .
- ▶ Therefore,  $h \circ Q_N$  maps an open singleton onto a non-open singleton.
- ▶ Therefore,  $h \circ Q_N$  is not an open map.
- ▶ Therefore,  $Q_N$  is not an open map.

# Open generation

## Theorem (Ščepin, essentially)

Given a compactum  $X$ , the following are equivalent.

1.  $Q_N: X \rightarrow X/N$  is open for all  $N \prec H(\theta)$  with  $X \in N$ .
2.  $Q_N: X \rightarrow X/N$  is open for all **countable**  $N \prec H(\theta)$  with  $X \in N$ .
3.  $X$  has a distance function  $\rho(x, C)$  between points and regular closed sets that satisfies certain axioms. . .
4.  $X$  has a “capacity,” a precursor to  $\rho$  as above consisting of maps  $\varepsilon_B: B \rightarrow [0, 1]$  where  $B$  ranges over a base of  $X$  and. . .

Ščepin stated his results in terms of special kinds of inverse limits instead of elementary submodels.

## Definition

Say that a compactum  $X$  is **openly generated** (OG) if it satisfies the above conditions.

# Retracts of $2^\kappa$

## Definition

Say that a space is **Dugundji** if it is  $\cong$  a retract of a power of 2.

## Theorem (Ščepin)

- ▶ *All Dugundji spaces are OG.*
- ▶ *All OG Boolean spaces of weight  $\leq \omega_1$  are Dugundji.*
- ▶  *$2^{\omega_1}$  is the only Dugundji space with all points of character  $\omega_1$ .*

## Corollary

$$2^{\omega_1} \cong SP^n(2^{\omega_1}) \cong \exp(2^{\omega_1}).$$



## $d$ -ary open generation

### Lemma

Given a compactum  $X$  and  $d < \omega$ , the following are equivalent.

- ▶  $\mathcal{Q}_{\bigcup_{i < d-1} N_i}$  is open for all  $N_0, \dots, N_{d-1} \prec H(\theta)$  with  $X \in N_i$ .
- ▶  $\mathcal{Q}_{\bigcup_{i < d-1} N_i}$  is open for all **countable**  $N_0, \dots, N_{d-1} \prec H(\theta)$  with  $X \in N_i$ .

### Definition

Say that a compactum  $X$  is  **$d$ -arily openly generated** ( $\text{OG}_d$ ) if it satisfies the above conditions.

Remarks:

- ▶ All compacta are  $\text{OG}_1$ .
- ▶  $\text{OG}_2$  is the same as  $\text{OG}$ .
- ▶ If  $n > m$ , then  $\text{OG}_n$  implies  $\text{OG}_m$ .

# A hierarchy theorem

## Theorem (Milovich)

1. *The Dugundji spaces are exactly the  $OG_{<\omega}$  boolean spaces.*
2. *All  $OG_{d+1}$  Boolean spaces of weight  $\leq \omega_d$  are Dugundji.*
3. *There is a Boolean space  $Y$  of weight  $\omega_d$  that is  $OG_d$  but not  $OG_{d+1}$ .*

(Our set-theoretic lemma about  $\omega_d$  is relevant to proving 2 and 3.)

## Corollary

$\exp(2^{\omega_2})$  and  $SP^n(2^{\omega_2})$  (for  $n \geq 2$ ) are  $OG_2$  but not  $OG_3$ .

## Question

For  $d \geq 3$ , the only known  $Y$  as above is an ad-hoc construction. Is there a “natural” space that is  $OG_3$  but not  $OG_4$ ?

# Inverse semilattice system

An inverse limit system

$$((X_i)_{i \in P}, (f_{i,j}: X_j \rightarrow X_i)_{i < j})$$

is an **inverse semilattice system** if:

$$\begin{array}{ccc} X_i & \longleftarrow & X_k \\ & \searrow f_{i,k} & \downarrow \\ & & X_j \\ & \swarrow f_{i \wedge j, i} & \swarrow f_{i \wedge j, j} \\ X_{i \wedge j} & \longleftarrow & X_j \end{array}$$

►  $P$  is a meet-semilattice.

►  $X_i \xrightarrow{f_{i \wedge j, i}} X_{i \wedge j} \xleftarrow{f_{i \wedge j, j}} X_j$  is the **colimit** of  $X_i \xleftarrow{f_{k,i}} X_k \xrightarrow{f_{k,j}} X_j$  for all  $i, j < k$ .

$P$  is a **meet-semilattice** if every pair  $\{i, j\} \subseteq P$  has a greatest lower bound  $i \wedge j$ .

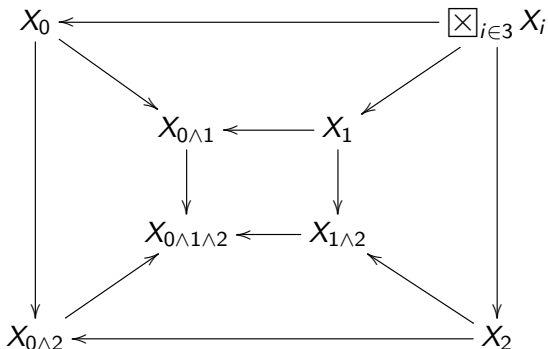
Don't worry about what "colimit" means.

## Higher-arity fiber products

Given  $d < \omega$ , an inverse semilattice system  $((X_i)_{i \in P}, \vec{f})$ , and  $D \in [P]^d$ , define the  $d$ -ary fiber product  $\boxtimes_{i \in D} X_i$ :

$$\boxtimes_{i \in D} X_i = \left\{ \vec{x} \in \prod_{i \in D} X_i : \forall i, j \in D \quad f_{i \wedge j, i}(x_i) = f_{i \wedge j, i}(x_j) \right\}.$$

Diagram for  $D = 3$ :



## Higher-arity properties

### Lemma

In an inverse semilattice system  $(\vec{X}, \vec{f})$ , if  $i_0, \dots, i_{n-1} < j$ , then the diagonal product  $\prod_{k < n} f_{i_k, j}$  maps  $X_j$  **into** the fiber product  $\boxtimes_{k < n} X_{i_k}$ .

### Definition

An inverse semilattice system  $(\vec{X}, \vec{f})$  is  **$n$ -surjective** if, for all  $i_0, \dots, i_{n-1} < j$ , the diagonal product  $\prod_{k < n} f_{i_k, j}$  maps  $X_j$  **onto** the fiber product  $\boxtimes_{k < n} X_{i_k}$ .

### Definition

An inverse semilattice system  $(\vec{X}, \vec{f})$  is  **$n$ -open** if, for all  $i_0, \dots, i_{n-1} < j$ , the diagonal product  $\prod_{k < n} f_{i_k, j}$  is an **open** map from  $X_j$  into the fiber product  $\boxtimes_{k < n} X_{i_k}$ .

# Retracts of $2^\kappa$ revisited

## Theorem (Milovich)

Given a boolean space  $X$ , following are equivalent.

- ▶  $X$  is Dugundji.
- ▶  $X$  is  $\cong$  the limit of a  $<\omega$ -surjective and  $<\omega$ -open inverse semilattice system  $(\vec{X}, \vec{f})$  of **second countable** spaces  $\vec{X}$ .
- ▶  $X$  is  $\cong$  the limit of a  $<\omega$ -surjective inverse semilattice system  $(\vec{X}, \vec{f})$  of **finite** spaces  $\vec{X}$ .

## $d$ -ary open generation revisited

### Theorem (Milovich)

Given  $1 \leq d < \omega$ , a compactum  $X$  is  $\text{OG}_d$  iff  $X$  is  $\cong$  the limit of a  $d$ -surjective and  $(d - 1)$ -open inverse semilattice system  $(\vec{X}, \vec{f})$  of second countable compacta  $\vec{X}$ .

### Corollary

A compactum is  $\text{OG}$  iff it is  $\cong$  the limit of a 2-surjective 1-open inverse semilattice system of second countable compacta  $\vec{X}$ .

### Theorem (Milovich)

However, there is an  $\text{OG}$  boolean space *not*  $\cong$  the limit of any 2-surjective inverse semilattice system of finite spaces  $X$ .