Higher-arity properties of inverse limit systems

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These slides are online: `http://dkmj.org/academic`

Basic terminology and scope

compactum = compact Hausdorff space

Boolean space = compactum with a clopen base

dyadic = continuous image of a power of 2

crowded = without isolated points

ccc = every pairwise disjoint family of open sets is countable.

- Most of our example spaces will be boolean and dyadic.
- All of our example spaces will be crowded ccc compacta.

open map = function that maps open sets onto open sets

Some notation

$X \cong Y$ means X is homeomorphic to Y.

π_0 and π_1 are the first and second coordinate projections:

$$\pi_0(x, y) = x \text{ and } \pi_1(x, y) = y$$

$h = f \times g$ means h is the diagonal product of f and g:

$$h(x) = (f(x), g(x))$$

$Z = f \boxtimes g$ means Z is the fiber product of f and g:

$$Z = \{(x, y) : f(x) = g(y)\}$$

Prelude: symmetric powers of 2^κ

- 2^κ is space of all functions from κ to 2.
- $(2^\kappa)^n$ is space of all functions from n to 2^κ.
- $SP^n(2^\kappa)$ is the quotient space of $(2^\kappa)^n$ induced by:

  $$f \sim g \text{ iff } f = g \circ \tau \text{ for some permutation } \tau\colon n \to n$$
- Given $0 < m < n < \omega \leq \kappa$, we have $(2^\kappa)^m \cong (2^\kappa)^n$ since $|\kappa \times m| = |\kappa \times n|$.
- Is $SP^m(2^\kappa) \cong SP^n(2^\kappa)$?

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- If $\kappa = \omega$, then $SP^m(2^\kappa) \cong SP^n(2^\kappa) \cong 2^\kappa$ simply because 2^ω is up to homeomorphism the only second countable crowded Boolean space.

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- (Ščepin) If $\kappa \geq \omega_2$ then $SP^m(2^\kappa) \ncong SP^n(2^\kappa)$.

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- (Ščepin) If $\kappa = \omega_1$ then again $SP^m(2^\kappa) \cong SP^n(2^\kappa) \cong 2^\kappa$. However, the proof is a lot harder.
- (Ščepin) If $\kappa \geq \omega_2$ then $SP^m(2^\kappa) \not\cong SP^n(2^\kappa)$. In fact, Ščepin's proof shows that $SP^n(2^\kappa)$ is not a retract of any homeomorphic copy of $SP^m(2^\kappa)$.

So, why isn't, say, $SP^7(2^{\omega_2})$ a retract of $SP^6(2^{\omega_2})$?

Lemma

Given a cardinal κ, the following are equivalent:

- $\kappa \geq \omega_2$.
- *There exist $M, N \prec H(\theta)$ such that $M \cap \kappa \not\subseteq N \cap \kappa \not\subseteq M \cap \kappa$.*
- *There exist countable $M, N \prec H(\theta)$ such that $M \cap \kappa \not\subseteq N \cap \kappa \not\subseteq M \cap \kappa$.*

Definition

"$M \prec H(\theta)$" implies that M satisfies arbitrary finite lists of finitary closure properties that I don't want to write down. More precisely:

- $M \prec H(\theta)$ means $(M, \in\restriction M, \sqsubset\restriction M)$ is a first-order elementary submodel of a set-theoretic universe well-ordered by $\sqsubset$.
- To avoid going beyond ZFC, choose a "universe" of the form $(H(\theta), \in, \sqsubset)$ for θ a regular cardinal large enough that $H(\theta)$ has all the power sets we need for the argument at hand.

An impossible fiber retract

Let $$\begin{tikzcd} SP^7(2^{\omega_2}) \arrow[r, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & SP^6(2^{\omega_2}) \arrow[r, "g"'] & SP^7(2^{\omega_2}) \end{tikzcd}.$$

Let $N_0, N_1 \prec H(\theta)$ with $f, g \in N_i$ and $E_i = N_i \cap \omega_2 \not\subseteq E_{1-i}$.

Let $Y_j = SP^j(2^{E_0 \cup E_1})$ for each $j \in \{6, 7\}$.

Let Z_j be the fiber product $\pi^{E_0}_{j,E_0\cap E_1} \boxtimes \pi^{E_1}_{j,E_0\cap E_1}$

where $\pi^B_{j,A} \colon SP^j(2^B) \to SP^j(2^A)$, $f/{\sim} \mapsto ((f_i \restriction A)_{i<j})/{\sim}$.

Then the diagonal product $\zeta_j = \pi^{E_0\cup E_1}_{j,E_0} \times \pi^{E_0\cup E_1}_{j,E_1}$ maps Y_j into Z_j.

$$\begin{tikzcd} Y_7 \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "\zeta_7"] & Y_6 \arrow[r] \arrow[d, "\zeta_6"'] & Y_7 \arrow[d, "\zeta_7"'] \\ Z_7 \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & Z_6 \arrow[r] & Z_7 \end{tikzcd}$$

- ζ_7 has a fiber of size 7!.
- ζ_6 has no fibers larger than 6!.
- Contradiction!

Why are there fibers of size 7!?

Let $a \in (2^{E_0 \cap E_1})^7$ and $a_i = a_j$ for all $i < j < 7$.
Let $b \in (2^{E_0 \setminus E_1})^7$ and $b_i \neq b_j$ for all $i < j < 7$.
Let $c \in (2^{E_1 \setminus E_0})^7$ and $c_i \neq c_j$ for all $i < j < 7$.
Then, for all permutations $\rho, \sigma \colon 7 \to 7$:

$$\begin{aligned}
\zeta_7((a_i \cup b_{\rho(i)} \cup c_{\sigma(i)})_{i<7}/{\sim}) &= \zeta_7((a_i \cup b_i \cup c_{\tau(i)})_{i<7}/{\sim}) \\
&= ((a_i \cup b_i)_{i<7}/{\sim}, (a_i \cup c_{\tau(i)})_{i<7}/{\sim}) \\
&= ((a_i \cup b_i)_{i<7}/{\sim}, (a_i \cup c_i)_{i<7}/{\sim})
\end{aligned}$$

where $\tau = \sigma \circ \rho^{-1}$.

$$\begin{tikzcd}
 & & Y_7 \arrow[dll, bend right, "\pi^{E_0 \cup E_1}_{7,E_0}"'] \arrow[dl, "\zeta_7"] \arrow[dl, bend left, shift left=4, "\pi^{E_0 \cup E_1}_{7,E_1}"] \\
SP^7(2^{E_0}) \arrow[d, "\pi^{E_0}_{7,E_0 \cap E_1}"'] & Z_7 \arrow[l] \arrow[d] & \\
SP^7(2^{E_0 \cap E_1}) & SP^7(2^{E_1}) \arrow[l, "\pi^{E_1}_{7,E_0 \cap E_1}"] &
\end{tikzcd}$$

What new symmetries break at ω_3?

“Symmetry breaking” at ω_2 is not just for symmetric powers. For example, 2^{ω_1} is homeomorphic to its Vietoris hyperspace $\exp(2^{\omega_1})$ (Ščepin) but $\exp(2^{\omega_2})$ is not a continuous image of 2^{ω_2} (Shapiro).

Are there analogous phenomena at ω_3?
Bell's non-supercompact dyadic compactum of weight ω_3 is loosely analogous. However, we don't know if ω_3 is least possible.
At least our lemma about ω_2 has a clear analog:

Lemma

Let $0 < d < \omega$ and let κ be a cardinal. The following are equivalent.

- $\kappa \geq \omega_d$.
- *There exist* $N_0, \ldots, N_{d-1} \prec H(\theta)$ *such that* $\kappa \cap \bigcap_{j \neq i} N_j \not\subseteq N_i$ *for all* $i < d$.
- *There exist countable* $N_0, \ldots, N_{d-1} \prec H(\theta)$ *such that* $\kappa \cap \bigcap_{j \neq i} N_j \not\subseteq N_i$ *for all* $i < d$.

Quotients induced by elementary submodels

Suppose that X is $T_{3.5}$, I is a set, and $X \in N_i \prec H(\theta)$ for all $i \in I$.
(By "$X \in N_i$" we actually mean "the topology of X is an element of N_i.")

Definition

$p \not\sim q$ if $f(p) \neq f(q)$ for some $f \in C(X, \mathbb{R}) \cap \bigcup_{i \in I} N_i$.
Let $X / \bigcup_{i \in I} N_i$ be the quotient space induced by $\sim$.
Let $\mathcal{Q}_{\bigcup_{i \in I} N_i}$ be the associated quotient map.

Lemma

If X is a compactum, or even just T_4, then the following are equivalent:

- $p \not\sim q$
- *p and q have disjoint closed neighborhoods $U, V \in N_i$ for some i.*
- *p and q have disjoint closed G_δ neighborhoods $U \in N_i$ and $V \in N_j$ for some i, j.*

Open quotient maps

When is the quotient map $\mathcal{Q}_{\mathcal{S}}$ an open map?

Example

Given $X = \omega_1 + 1$ and a countable $N \prec H(\theta)$, we have:

- There is a natural homeomorphism $h\colon X/N \cong \delta + 1$ where $\delta = \omega_1 \cap N$:

$$h(\mathcal{Q}_N(\alpha)) = \begin{cases} \alpha & \text{if } \alpha < \delta \\ \delta & \text{if } \alpha \geq \delta \end{cases}$$

- $h \circ \mathcal{Q}_N$ sends the isolated point $\delta + 1$ to the limit point δ.
- Therefore, $h \circ \mathcal{Q}_N$ maps an open singleton onto a non-open singleton.
- Therefore, $h \circ \mathcal{Q}_N$ is not an open map.
- Therefore, $\mathcal{Q}_N$ is not an open map.

Open generation

Theorem (Ščepin, essentially)

Given a compactum X, the following are equivalent.

1. $\mathcal{Q}_N \colon X \to X/N$ *is open for all* $N \prec H(\theta)$ *with* $X \in N$.
2. $\mathcal{Q}_N \colon X \to X/N$ *is open for all countable* $N \prec H(\theta)$ *with* $X \in N$.
3. X *has a distance function* $\rho(x, C)$ *between points and regular closed sets that satisfies certain axioms...*
4. X *has a "capacity," a precursor to* ρ *as above consisting of maps* $\varepsilon_B \colon B \to [0, 1]$ *where* B *ranges over a base of* X *and...*

Ščepin stated his results in terms of special kinds of inverse limits instead of elementary submodels.

Definition

Say that a compactum X is openly generated (OG) if it satisfies the above conditions.

Retracts of 2^κ

Definition

Say that a space is Dugundji if it is $\cong$ a retract of a power of 2.

Theorem (Ščepin)

- *All Dugundji spaces are* OG.
- *All* OG *Boolean spaces of weight* $\leq \omega_1$ *are Dugundji.*
- 2^{ω_1} *is the only Dugundji space with all points of character* ω_1.

Corollary

$2^{\omega_1} \cong SP^n(2^{\omega_1}) \cong \exp(2^{\omega_1})$.

d-ary open generation

Lemma

Given a compactum X and $d < \omega$, the following are equivalent.

- $\mathcal{Q}_{\bigcup_{i<d-1} N_i}$ *is open for all* $N_0, \ldots, N_{d-1} \prec H(\theta)$ *with* $X \in N_i$.
- $\mathcal{Q}_{\bigcup_{i<d-1} N_i}$ *is open for all countable* $N_0, \ldots, N_{d-1} \prec H(\theta)$ *with* $X \in N_i$.

Definition

Say that a compactum X is d-arily openly generated (OG_d) if it satisfies the above conditions.

Remarks:

- All compacta are OG_1.
- OG_2 is the same as OG.
- If $n > m$, then OG_n implies OG_m.

A hierarchy theorem

Theorem (Milovich)

1. *The Dugundji spaces are exactly the* $\mathrm{OG}_{<\omega}$ *boolean spaces.*
2. *All* OG_{d+1} *Boolean spaces of weight* $\leq \omega_d$ *are Dugundji.*
3. *There is a Boolean space* Y *of weight* ω_d *that is* OG_d *but not* OG_{d+1}*.*

(Our set-theoretic lemma about ω_d is relevant to proving 2 and 3.)

Corollary

$\exp(2^{\omega_2})$ *and* $SP^n(2^{\omega_2})$ *(for* $n \geq 2$*) are* OG_2 *but not* OG_3*.*

Question

For $d \geq 3$, the only known Y as above is an ad-hoc construction.
Is there a "natural" space that is OG_3 but not OG_4?

Inverse semilattice system

$$\begin{tikzcd}
X_i \arrow[d, "f_{i\wedge j,i}"] & X_k \arrow[l, "f_{i,k}"] \arrow[d, "f_{j,k}"'] \\
X_{i\wedge j} & X_j \arrow[l, "f_{i\wedge j,j}"']
\end{tikzcd}$$

An inverse limit system

$$((X_i)_{i\in P}, (f_{i,j} \colon X_j \to X_i)_{i<j})$$

is an inverse semilattice system if:

- P is a meet-semilattice.
- $X_i \xrightarrow{f_{i\wedge j,i}} X_{i\wedge j} \xleftarrow{f_{i\wedge j,j}} X_j$ is the colimit of $X_i \xleftarrow{f_{k,i}} X_k \xrightarrow{f_{k,j}} X_j$ for all $i, j < k$.

P is a meet-semilattice if every pair $\{i, j\} \subseteq P$ has a greatest lower bound $i \wedge j$.

Don't worry about what "colimit" means.

Higher-arity fiber products

Given $d < \omega$, an inverse semilattice system $((X_i)_{i \in P}, \vec{f})$, and $D \in [P]^d$, define the d-ary fiber product $\boxtimes_{i \in D} X_i$:

$$\mathop{\boxtimes}\limits_{i \in D} X_i = \left\{ \vec{x} \in \prod_{i \in D} X_i : \forall i, j \in D \quad f_{i \wedge j, i}(x_i) = f_{i \wedge j, i}(x_j) \right\}.$$

Diagram for $D = 3$:

$$\begin{tikzcd}
X_0 \arrow[dr] \arrow[ddd] & & & \boxtimes_{i \in 3} X_i \arrow[lll] \arrow[dl] \arrow[ddd] \\
& X_{0 \wedge 1} \arrow[d] & X_1 \arrow[l] \arrow[d] & \\
& X_{0 \wedge 1 \wedge 2} & X_{1 \wedge 2} \arrow[l] & \\
X_{0 \wedge 2} \arrow[ur] & & & X_2 \arrow[lll] \arrow[ul]
\end{tikzcd}$$

Higher-arity properties

Lemma

In an inverse semilattice system $(\vec{X}, \vec{f})$, if $i_0, \ldots, i_{n-1} < j$, then the diagonal product $\prod_{k<n} f_{i_k,j}$ maps X_j into the fiber product $\boxtimes_{k<n} X_{i_k}$.

Definition

An inverse semilattice system $(\vec{X}, \vec{f})$ is *n*-surjective if, for all $i_0, \ldots, i_{n-1} < j$, the diagonal product $\prod_{k<n} f_{i_k,j}$ maps X_j onto the fiber product $\boxtimes_{k<n} X_{i_k}$.

Definition

An inverse semilattice system $(\vec{X}, \vec{f})$ is *n*-open if, for all $i_0, \ldots, i_{n-1} < j$, the diagonal product $\prod_{k<n} f_{i_k,j}$ is an open map from X_j into the fiber product $\boxtimes_{k<n} X_{i_k}$.

Retracts of 2^κ revisited

Theorem (Milovich)

Given a boolean space X, following are equivalent.

- *X is Dugundji.*
- *X is $\cong$ the limit of a $<\omega$-surjective and $<\omega$-open inverse semilattice system $(\vec{X}, \vec{f})$ of second countable spaces $\vec{X}$.*
- *X is $\cong$ the limit of a $<\omega$-surjective inverse semilattice system $(\vec{X}, \vec{f})$ of finite spaces $\vec{X}$.*

d-ary open generation revisited

Theorem (Milovich)

Given $1 \leq d < \omega$, a compactum X is OG_d iff X is $\cong$ the limit of a d-surjective and $(d-1)$-open inverse semilattice system $(\vec{X}, \vec{f})$ of second countable compacta $\vec{X}$.

Corollary

A compactum is OG iff it is $\cong$ the limit of a 2-surjective 1-open inverse semilattice system of second countable compacta $\vec{X}$.

Theorem (Milovich)

However, there is an OG boolean space not $\cong$ the limit of any 2-surjective inverse semilattice system of finite spaces X.