Higher-arity properties of inverse limit systems

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Basic terminology and scope

compactum = compact Hausdorff space

Boolean space = compactum with a clopen base

dyadic = continuous image of a power of 2

crowded = without isolated points

ccc = every pairwise disjoint family of open sets is countable.

- Most of our example spaces will be boolean and dyadic.
- All of our example spaces will be crowded ccc compacta.

open map = function that maps open sets onto open sets

Some notation

$X \cong Y$ means X is homeomorphic to Y.

π_0 and π_1 are the first and second coordinate projections:

$$\pi_0(x, y) = x \text{ and } \pi_1(x, y) = y$$

$h = f \times g$ means h is the diagonal product of f and g:

$$h(x) = (f(x), g(x))$$

$Z = f \boxtimes g$ means Z is the fiber product of f and g:

$$Z = \{(x, y) : f(x) = g(y)\}$$

Prelude: symmetric powers of 2^κ

- 2^κ is space of all functions from κ to 2.
- $(2^\kappa)^n$ is space of all functions from n to 2^κ.
- $SP^n(2^\kappa)$ is the quotient space of $(2^\kappa)^n$ induced by:

$$f \sim g \text{ iff } f = g \circ \tau \text{ for some permutation } \tau\colon n \to n$$

- Given $0 < m < n < \omega \leq \kappa$, we have $(2^\kappa)^m \cong (2^\kappa)^n$ since $|\kappa \times m| = |\kappa \times n|$.
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- If $\kappa = \omega$, then $SP^m(2^\kappa) \cong SP^n(2^\kappa) \cong 2^\kappa$ simply because 2^ω is up to homeomorphism the only second countable crowded Boolean space.

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- (Ščepin) If $\kappa \geq \omega_2$ then $SP^m(2^\kappa) \not\cong SP^n(2^\kappa)$.
  In fact, Ščepin's proof shows that $SP^n(2^\kappa)$ is not a retract of any homeomorphic copy of $SP^m(2^\kappa)$.

So, why isn't, say, $SP^7(2^{\omega_2})$ a retract of $SP^6(2^{\omega_2})$?

Lemma

Given a cardinal κ, the following are equivalent:

- $\kappa \geq \omega_2$.
- *There exist $M, N \prec H(\theta)$ such that $M \cap \kappa \not\subseteq N \cap \kappa \not\subseteq M \cap \kappa$.*
- *There exist countable $M, N \prec H(\theta)$ such that $M \cap \kappa \not\subseteq N \cap \kappa \not\subseteq M \cap \kappa$.*

Definition

"$M \prec H(\theta)$" implies that M satisfies arbitrary finite lists of finitary closure properties that I don't want to write down. More precisely:

- $M \prec H(\theta)$ means $(M, \in\restriction M, \sqsubset\restriction M)$ is a first-order elementary submodel of a set-theoretic universe well-ordered by $\sqsubset$.
- To avoid going beyond ZFC, choose a "universe" of the form $(H(\theta), \in, \sqsubset)$ for θ a regular cardinal large enough that $H(\theta)$ has all the power sets we need for the argument at hand.

An impossible fiber retract

Let $\overbrace{SP^7(2^{\omega_2}) \xrightarrow[f]{} SP^6(2^{\omega_2}) \xrightarrow[g]{} SP^7(2^{\omega_2})}^{\mathrm{id}}$.

Let $N_0, N_1 \prec H(\theta)$ with $f, g \in N_i$ and $E_i = N_i \cap \omega_2 \not\subseteq E_{1-i}$.

Let $Y_j = SP^j(2^{E_0 \cup E_1})$ for each $j \in \{6, 7\}$.

Let Z_j be the fiber product $\pi^{E_0}_{j,E_0 \cap E_1} \boxtimes \pi^{E_1}_{j,E_0 \cap E_1}$

where $\pi^B_{j,A} \colon SP^j(2^B) \to SP^j(2^A)$, $f/{\sim} \mapsto ((f_i \restriction A)_{i<j})/{\sim}$.

Then the diagonal product $\zeta_j = \pi^{E_0 \cup E_1}_{j,E_0} \times \pi^{E_0 \cup E_1}_{j,E_1}$ maps Y_j into Z_j.

$$\begin{array}{ccccc} & & \mathrm{id} & & \\ Y_7 & \longrightarrow & Y_6 & \longrightarrow & Y_7 \\ \downarrow \zeta_7 & & \zeta_6 \downarrow & & \zeta_7 \downarrow \\ Z_7 & \longrightarrow & Z_6 & \longrightarrow & Z_7 \\ & & \mathrm{id} & & \end{array}$$

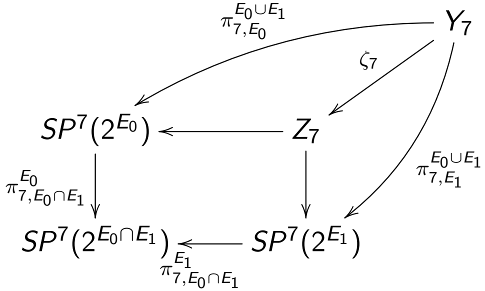
- ζ_7 has a fiber of size 7!.
- ζ_6 has no fibers larger than 6!.
- Contradiction!

Why are there fibers of size 7!?

Let $a \in (2^{E_0 \cap E_1})^7$ and $a_i = a_j$ for all $i < j < 7$.
Let $b \in (2^{E_0 \setminus E_1})^7$ and $b_i \neq b_j$ for all $i < j < 7$.
Let $c \in (2^{E_1 \setminus E_0})^7$ and $c_i \neq c_j$ for all $i < j < 7$.
Then, for all permutations $\rho, \sigma \colon 7 \to 7$:

$$\begin{aligned}
\zeta_7((a_i \cup b_{\rho(i)} \cup c_{\sigma(i)})_{i<7}/{\sim}) &= \zeta_7((a_i \cup b_i \cup c_{\tau(i)})_{i<7}/{\sim}) \\
&= ((a_i \cup b_i)_{i<7}/{\sim}, (a_i \cup c_{\tau(i)})_{i<7}/{\sim}) \\
&= ((a_i \cup b_i)_{i<7}/{\sim}, (a_i \cup c_i)_{i<7}/{\sim})
\end{aligned}$$

where $\tau = \sigma \circ \rho^{-1}$.



What new symmetries break at ω_3?

"Symmetry breaking" at ω_2 is not just for symmetric powers. For example, 2^{ω_1} is homeomorphic to its Vietoris hyperspace $\exp(2^{\omega_1})$ (Ščepin) but $\exp(2^{\omega_2})$ is not a continuous image of 2^{ω_2} (Shapiro).

Are there analogous phenomena at ω_3?
Bell's non-supercompact dyadic compactum of weight ω_3 is loosely analogous. However, we don't know if ω_3 is least possible.
At least our lemma about ω_2 has a clear analog:

Lemma

Let $0 < d < \omega$ and let κ be a cardinal. The following are equivalent.

- $\kappa \geq \omega_d$.
- *There exist $N_0, \ldots, N_{d-1} \prec H(\theta)$ such that $\kappa \cap \bigcap_{j \neq i} N_j \not\subseteq N_i$ for all $i < d$.*
- *There exist countable $N_0, \ldots, N_{d-1} \prec H(\theta)$ such that $\kappa \cap \bigcap_{j \neq i} N_j \not\subseteq N_i$ for all $i < d$.*

Quotients induced by elementary submodels

Suppose that X is $T_{3.5}$, I is a set, and $X \in N_i \prec H(\theta)$ for all $i \in I$.
(By "$X \in N_i$" we actually mean "the topology of X is an element of N_i.")

Definition

$p \not\sim q$ if $f(p) \neq f(q)$ for some $f \in C(X, \mathbb{R}) \cap \bigcup_{i \in I} N_i$.
Let $X / \bigcup_{i \in I} N_i$ be the quotient space induced by $\sim$.
Let $\mathcal{Q}_{\bigcup_{i \in I} N_i}$ be the associated quotient map.

Lemma

If X is a compactum, or even just T_4, then the following are equivalent:

- $p \not\sim q$
- *p and q have disjoint closed neighborhoods $U, V \in N_i$ for some i.*
- *p and q have disjoint closed G_δ neighborhoods $U \in N_i$ and $V \in N_j$ for some i, j.*

The view from the cube

Given X completely regular, $\mathcal{S} = \bigcup_{i \in I} N_i$, and $X \in N_i \prec H(\theta)$, the quotient map $\mathcal{Q}_{\mathcal{S}} \colon X \to X/\mathcal{S}$ is isomorphic to a restriction $p_{\mathcal{S}}$ of a coordinate projection:

- Let $\mathcal{C} = C(X, [0,1])$ and let $e \colon X \cong Y \subset [0,1]^{\mathcal{C}}$ be the Čech embedding $e(x)(f) = f(x)$.
- Define $\pi_{\mathcal{S}} \colon [0,1]^{\mathcal{C}} \to [0,1]^{\mathcal{C} \cap \mathcal{S}}$ by $y \mapsto y \restriction (\mathcal{C} \cap \mathcal{S})$.
- Let $Z = \pi_{\mathcal{S}}[Y]$ and $p_{\mathcal{S}} = \pi_{\mathcal{S}} \restriction Y$.
- Then there is a unique $h \colon X/\mathcal{S} \cong Z$ such that $p_{\mathcal{S}} \circ e = h \circ \mathcal{Q}_{\mathcal{S}}$.

$$\begin{array}{ccccc} X & \xrightarrow{e} & Y & \xrightarrow{\mathrm{id}} & [0,1]^{\mathcal{C}} \\ \downarrow{\scriptstyle \mathcal{Q}_{\mathcal{S}}} & & \downarrow{\scriptstyle p_{\mathcal{S}}} & & \downarrow{\scriptstyle \pi_{\mathcal{S}}} \\ X/\mathcal{S} & \xrightarrow[h]{} & Z & \xrightarrow[\mathrm{id}]{} & [0,1]^{\mathcal{C} \cap \mathcal{S}} \end{array}$$

Warning: Although coordinate projections such as $\pi_{\mathcal{S}}$ are open maps, restrictions of them such as $p_{\mathcal{S}}$ may not be open maps.

Open quotient maps

When is the quotient map $\mathcal{Q}_{\mathcal{S}}$ an open map?

Example

Given $X = \omega_1 + 1$ and a countable $N \prec H(\theta)$, we have:

- There is a natural homeomorphism $h\colon X/N \cong \delta + 1$ where $\delta = \omega_1 \cap N$:

$$h(\mathcal{Q}_N(\alpha)) = \begin{cases} \alpha & \text{if } \alpha < \delta \\ \delta & \text{if } \alpha \geq \delta \end{cases}$$

- $h \circ \mathcal{Q}_N$ sends the isolated point $\delta + 1$ to the limit point δ.
- Therefore, $h \circ \mathcal{Q}_N$ maps an open singleton onto a non-open singleton.
- Therefore, $h \circ \mathcal{Q}_N$ is not an open map.
- Therefore, $\mathcal{Q}_N$ is not an open map.

Open generation

Theorem (Ščepin, essentially)

Given a compactum X, the following are equivalent.

1. $\mathcal{Q}_N \colon X \to X/N$ *is open for all* $N \prec H(\theta)$ *with* $X \in N$.
2. $\mathcal{Q}_N \colon X \to X/N$ *is open for all countable* $N \prec H(\theta)$ *with* $X \in N$.
3. X *has a distance function* $\rho(x, C)$ *between points and regular closed sets that satisfies certain axioms. . .*
4. X *has a "capacity," a precursor to* ρ *as above consisting of maps* $\varepsilon_B \colon B \to [0, 1]$ *where* B *ranges over a base of* X *and. . .*

Ščepin stated his results in terms of special kinds of inverse limits instead of elementary submodels.

Definition

Say that a compactum X is openly generated (OG) if it satisfies the above conditions.

Some properties of openly generated compacta

- (Ščepin) Products of OG compacta are OG.
- (Ščepin) Symmetric powers of OG compacta are OG.
- (Ščepin) G_δ subspaces of OG compacta are OG.
- (Ščepin) Vietoris hyperspaces of OG compacta are OG.
- (Shapiro) Some OG compacta are not dyadic.
- (Ščepin) But, like a dyadic compactum, if X is a continuous image of an OG compactum, then X is ccc and has weight equal to its π-character.
- (Ščepin) Also like dyadic spaces, regular closed sets in OG compacta are G_δ.

Question (Ščepin)

If every OG compactum a continuous image of an OG boolean space?

Retracts of 2^κ

Definition

Say that a space is Dugundji if it is $\cong$ a retract of a power of 2.

Theorem (Ščepin)

- *All Dugundji spaces are* OG.
- *All* OG *Boolean spaces of weight* $\leq \omega_1$ *are Dugundji.*
- 2^{ω_1} *is the only Dugundji space with all points of character* ω_1.

Corollary

$2^{\omega_1} \cong SP^n(2^{\omega_1}) \cong \exp(2^{\omega_1})$.

d-ary open generation

Lemma

Given a compactum X and $d < \omega$, the following are equivalent.

- $\mathcal{Q}_{\bigcup_{i<d-1} N_i}$ *is open for all* $N_0, \ldots, N_{d-1} \prec H(\theta)$ *with* $X \in N_i$.
- $\mathcal{Q}_{\bigcup_{i<d-1} N_i}$ *is open for all countable* $N_0, \ldots, N_{d-1} \prec H(\theta)$ *with* $X \in N_i$.

Definition

Say that a compactum X is d-arily openly generated (OG_d) if it satisfies the above conditions.

Remarks:

- All compacta are OG_1.
- OG_2 is the same as OG.
- If $n > m$, then OG_n implies OG_m.

A hierarchy theorem

Theorem (Milovich)

1. *The Dugundji spaces are exactly the* $\mathrm{OG}_{<\omega}$ *boolean spaces.*
2. *All* OG_{d+1} *Boolean spaces of weight* $\leq \omega_d$ *are Dugundji.*
3. *There is a Boolean space* Y *of weight* ω_d *that is* OG_d *but not* OG_{d+1}*.*

(Our set-theoretic lemma about ω_d is relevant to proving 2 and 3.)

Corollary

$\exp(2^{\omega_2})$ *and* $SP^n(2^{\omega_2})$ *(for* $n \geq 2$*) are* OG_2 *but not* OG_3*.*

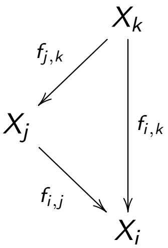
Question

For $d \geq 3$, the only known Y as above is an ad-hoc construction.
Is there a "natural" space that is OG_3 but not OG_4?

Inverse diagrams

Definition

- An inverse diagram is a pair $((X_i)_{i \in P}, (f_{i,j})_{i<j})$ where:
  - $(P, <)$ is a strict partial order.
  - X_i is a compactum.
  - $f_{i,j} \colon X_j \to X_i$ is a quotient map. (Between compacta, all continuous surjections are quotient maps.)
  - $f_{i,k} = f_{i,j} \circ f_{j,k}$ if $i < j < k$.
  - (Define $f_{i,i} = id \colon X_i \to X_i$.)



- The canonical limit of $(\vec{X}, \vec{f})$ is the compactum $X_\infty = \{\vec{x} \in \prod_i X_i : \forall (i < j)\ x_i = f_{i,j}(x_j)\}$ together with maps $f_{i,\infty} \colon X_\infty \to X_i$, $\vec{x} \mapsto x_i$.

Inverse limit systems

Definition

- $(P, <)$ is directed if all its finite subsets have upper bounds.
- If P is directed, we call $((X_i)_{i\in P}, (f_{i,j})_{i<j})$ an inverse limit system and X_∞ an inverse limit.

Warning: If P is not directed then $f_{i,\infty}$ may not be surjective and X_∞ may even be empty.

Retracts of 2^κ as linear inverse limits

Inverse diagrams have been used to state and prove many internal structural characterizations of properties of compacta.

Theorem (Koppelberg)

A space X is Dugundji iff X is $\cong$ the limit of some $((X_\alpha)_{\alpha<\kappa}, (f_{\alpha,\beta})_{\alpha<\beta<\kappa})$ where, for all α:

- $X_0 = 1$.
- $X_{\alpha+1} = K_0 \oplus K_1$ *where* $\{K_0, K_1\}$ *is a clopen cover of* X_α.
- $f_{\alpha,\alpha+1}(x, i) = x$.
- *For limit* γ, $(X_\gamma, (f_{\alpha,\gamma})_{\alpha<\gamma}) = \lim((X_\alpha)_{\alpha<\gamma}, (f_{\alpha,\beta})_{\alpha<\beta<\gamma})$.

For any $(\vec{X}, \vec{f})$ as above, all maps $f_{\alpha,\beta}$ and $f_{\alpha,\infty}$ are open.

Colimits and inverse semilattices

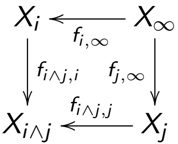
Definition

- The canonical colimit of an inverse limit system $((X_i)_{i\in P}, (f_{i,j})_{i<j})$ is the compactum $X_{-\infty} = X_\infty/\sim$ where $\sim$ is the intersection of all closed equivalence relations that contain $\bigcup_{i\in P}\{(x,y) : f_{i,\infty}(x) = f_{i,\infty}(y)\}$.
- The maps associated with $X_{-\infty}$ are $f_{-\infty,i}\colon X_i \to X_{-\infty}$ where $p \mapsto [\vec{x}]$ where $x_i = p$.

Definition

$(\vec{X}, \vec{f})$ is an inverse semilattice if:

- P is a directed meet-semilattice.
- $(X_{i\wedge j}, (f_{i\wedge j,i}, f_{i\wedge j,j}))$ is the colimit of $((X_i, X_j, X_\infty), (f_{i,\infty}, f_{j,\infty}))$.

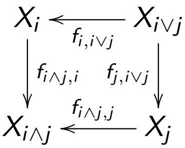


P is a meet-semilattice if every pair $\{i,j\} \subseteq P$ has a greatest lower bound $i \wedge j$.

Retracts of 2^κ as nonlinear inverse limits

Call $((X_i)_{i\in P}, \vec{f})$ an inverse lattice if:

- P is a lattice.
- $(X_{i\wedge j}, (f_{i\wedge j,i}, f_{i\wedge j,j}))$ is the colimit of $((X_i, X_j, X_{i\vee j}), (f_{i,i\vee j}, f_{j,i\vee j}))$.
- $(X_{i\vee j}, (f_{i,i\vee j}, f_{j,i\vee j}))$ is the limit of $((X_i, X_j, X_{i\wedge j}), (f_{i\wedge j,i}, f_{i\wedge j,j}))$.



P is a lattice if every pair $\{i, j\} \subseteq P$ has a greatest lower bound $i \wedge j$ and least upper bound $i \vee j$.

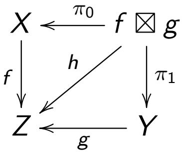
Theorem (Milovich)

Given a boolean space X, following are equivalent.

- *X is Dugundji.*
- *X is $\cong$ the limit of an inverse lattice of second countable spaces with open maps.*
- *X is $\cong$ the limit of an inverse lattice of finite spaces.*

Fiber products make lattices

Given $X \xrightarrow{f} Z \xleftarrow{g} Y$, its limit is isomorphic to the fiber product $f \boxtimes g = \{(x, y) : f(x) = g(y)\}$ with three outgoing arrows π_0, π_1, and $h = f \circ \pi_0 = g \circ \pi_1$.



Moreover, Z with the three incoming arrows f, g, and h is isomorphic to the colimit of $X \xleftarrow{\pi_0} (f \boxtimes g) \xrightarrow{\pi_1} Y$.

Therefore, the above diagram is an inverse lattice.

In category theory, the above diagram called a pullback square.

Semilattice and lattice expansions

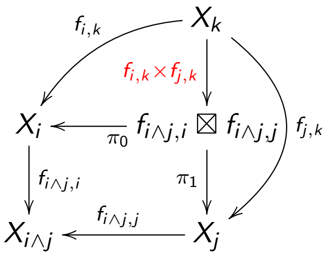
We can expand any inverse limit system $(\vec{X}, \vec{f})$ to an inverse semilattice by inserting the missing colimits.

However, we may not be able to expand $(\vec{X}, \vec{f})$ to an inverse lattice.

Why? If we insert a fiber product and the coordinate projections from this fiber product, then we are required to insert maps to this fiber product may not be quotient maps...

Binary obstructions

Given an inverse semilattice $(\vec{X}, \vec{f})$, every diagonal product of the form $f_{i,k} \times f_{j,k}$ continuously maps X_k into the fiber product $f_{i\wedge j,i} \boxtimes f_{i\wedge j,j}$ but may not map X_k onto this fiber product. Call instances of this problem binary obstructions.



Example

$S_7(2^{\omega_2}) \cong \lim((SP^7(2^C))_{C\in[P]^{\aleph_0}}, (\pi^D_{7,C})_{C\subsetneq D})$; there are no binary obstructions. In particular, $\zeta_7 \colon Z_7 \to Y_7$ is surjective.

Theorem (Milovich)

• *A compactum is* OG *iff it is* $\cong$ *the limit of an inverse semilattice of open quotient maps between second countable compacta without binary obstructions.*

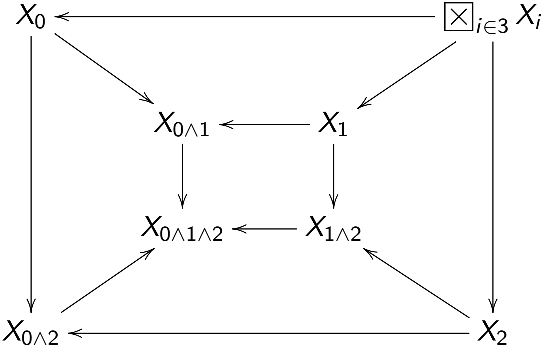
• *However, there is an* OG *boolean space not* $\cong$ *any limit of an inverse semilattice of finite spaces without binary obstructions.*

Pullback hypercubes

Given P a meet-semilattice, $d < \omega$, an inverse diagram $((X_i)_{i \in P}, \vec{f})$, and $D \in [P]^d$, define the d-ary fiber product $\boxtimes_{i \in D} X_i$:

$$\boxtimes_{i \in D} X_i = \left\{ \vec{x} \in \prod_{i \in D} X_i : \forall i, j \in D \quad f_{i \wedge j, i}(x_i) = f_{i \wedge j, i}(x_j) \right\}.$$

Diagram for $D = 3$:



Higher-arity obstructions

Definition

An inverse semilattice $(\vec{X}, \vec{f})$ has a d-ary obstruction if there exist distinct $i_0, \ldots, i_{d-1} < j$ such that the diagonal product $\prod_{k<d} f_{i_k,j}$ does not map X_j onto the fiber product $\boxtimes_{k<d} X_{i_k}$.

Lemma (Milovich)

Given $2 \leq d < \omega$, a compactum X is OG_d iff X is $\cong$ the limit of an inverse semilattice $(\vec{X}, \vec{f})$ such that

- *Each X_i is a second-countable compactum.*
- *$\prod_{k<d-1} f_{i_k,j}$ is open for all $i_0, \ldots, i_{d-2} < j$,*
- *$(\vec{X}, \vec{f})$ has no $\leq d$-ary obstructions.*

Proof strategy for $\mathrm{OG}_d \not\Rightarrow \mathrm{OG}_{d+1}$

- Build a strict partial ordering $\triangleleft$ that is directed and has countable lower cones.

- Simultaneously build an inverse limit system $((X_\alpha)_{\alpha<\omega_d}, (f_{\beta,\alpha})_{\beta\triangleleft\alpha})$.

- Ensure that if $\beta_0, \ldots, \beta_{d-1} \triangleleft \alpha$ then:
  - $\prod_{i<d} f_{\beta_i,\alpha} \colon X_\alpha \to \boxtimes_{i<d} X_{\beta_i}$ is onto.
  - $\prod_{i<d-1} f_{\beta_i,\alpha} \colon X_\alpha \to \boxtimes_{i<d-1} X_{\beta_i}$ is open.
  - $\prod_{i<d} f_{\beta_i,\alpha} \colon X_\alpha \to \boxtimes_{i<d} X_{\beta_i}$ is “usually” not open.

How to partially order ω_d

We will simultaneously construct $\lhd$ and sequences $(X_\alpha)_{\alpha<\omega_d}$, $(f_{\beta,\alpha})_{\beta\lhd\alpha}$, and $(M_\alpha)_{\alpha<\omega_d}$ such that:

- $X_0 = 2^\omega \cong X_\alpha$.
- M_α is $\prec H(\theta)$ and countable.
- The sequences $(X_\beta)_{\beta<\alpha+1}$, $(f_{\gamma,\beta})_{\gamma\lhd\beta<\alpha}$, and $(M_\beta)_{\beta<\alpha}$ are elements of M_α.
- $\{f_{\beta,\alpha} : \beta \lhd \alpha\}$ is a subset of M_α.

Define $\beta \lhd \alpha \Leftrightarrow M_\beta \in M_\alpha$ and note that

$$M_\beta \in M_\alpha \Leftrightarrow M_\beta \subsetneq M_\alpha \Leftrightarrow \beta \in \alpha \cap M_\alpha \Rightarrow \beta < \alpha.$$

Ignoring $\vec{X}$ and $\vec{f}$, a sequence $\vec{M}$ as above is called a Davies sequence or a long ω_1-approximation sequence. Such sequences have other applications in both set theory and topology. . .

Useful properties of $\vec{M}$

$\vec{M}$ is essentially a very very weak gap-d morass that exists in ZFC.

- $\lhd$ is directed and well-founded.
- Each lower cone $\{\beta : \beta \lhd \alpha\}$ partitions into $\leq d$-many countable directed sets $D_{\alpha,i}$ for $i < \daleth(\alpha)$.
- Each intersection $\bigcap_{\alpha \in I} M_\alpha$ is a directed union $\bigcup_{\alpha \in J} M_\alpha$.
  - Therefore, we won't have to think about colimits.

X_α and $f_{\beta,\alpha} \colon X_\alpha \to X_\beta$ are of the following form if $\alpha > 0$.

- $(Y_{\alpha,i}, (g_\beta)_{\beta \in D_{\alpha,i}}) = \lim((X_\beta)_{\beta \in D_{\alpha,i}}, \vec{f} \restriction D_{\alpha,i})$.
- $X_\alpha = C_0 \oplus C_1$ where $\{C_0, C_1\}$ is a closed cover of $\boxtimes_{i < \daleth(\alpha)} Y_{\alpha,i}$.
- $f_{\beta,\alpha}(\vec{y}, j) = g_\beta(y_i)$ for $\beta \in D_{\alpha,i}$.

Sufficient criteria for witnessing $\mathrm{OG}_d \not\Rightarrow \mathrm{OG}_{d+1}$

Again, X_α and $f_{\beta,\alpha}\colon X_\alpha \to X_\beta$ are of the following form if $\alpha > 0$.

- $(Y_{\alpha,i}, (g_\beta)_{\beta\in D_{\alpha,i}}) = \lim((X_\beta)_{\beta\in D_{\alpha,i}}, \vec{f} \restriction D_{\alpha,i})$.
- $X_\alpha = C_0 \oplus C_1$ where $\{C_0, C_1\}$ is a closed cover of $\boxtimes_{i<\daleth(\alpha)} Y_{\alpha,i}$.
- $f_{\beta,\alpha}(\vec{y}, j) = g_\beta(y_i)$ for $\beta \in D_{\alpha,i}$.

We additionally require:

1. $X_\alpha \ni (\vec{y}, j) \mapsto \vec{y} \restriction s$ defines an open map to $\boxtimes_{i\in s} Y_{\alpha,i}$ for all $s \in [\daleth(\alpha)]^{<d}$.
2. $X_\alpha \ni (\vec{y}, j) \mapsto y$ defines a non-open map if $\daleth(\alpha) = d$.

Finally:

- X_∞ will be OG_d thanks to 1.
- X_∞ will not be OG_{d+1} thanks to 2. More precisely, given any inverse limit system $(\vec{\Xi}, \vec{\xi})$, where $\Xi_\infty \cong X_\infty$ and each Ξ_i is second countable, we can use 2 to find $i_0, \ldots i_{d-1} < j$ such that $\prod_{k<d} \xi_{i_k,j}$ is not open.