Davies trees
and a paradox in the plane

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Mathematics and Physics Colloquium
Apr. 21, 2017

- By a *partial real function* we mean a set of points in the plane that passes the vertical line test:

  exactly one or zero points in each vertical line.
- Given a line L, by an *L-free set* we mean a set of points in the plane that passes the L test:

  exactly one or zero points in each line parallel to L.
- A set is L-free for some line L iff it is a rotated partial real function.
- By rotational invariance of area and the Fubini Theorem, if an L-free set C has a well-defined area at all (in the sense of Lebesgue measure), then it must have area zero.
- In particular, rotated continuous functions have area zero. More generally, subsets of rotated measurable functions have area zero.
- If each of sets $S_1, S_2, S_3, \ldots$ has area zero, then so does the union $\bigcup_{n=1}^{\infty} S_n$. (Proof: For each $\varepsilon > 0$, each S_n is covered by rectangles $R_{n,1}, R_{n,2}, R_{n,3}, \ldots$ with total area $< \varepsilon/2^n$.)
- Thus, given an L_n-free set C_n with well-defined area for each $n = 1, 2, 3, \ldots$, the union $\bigcup_{n=1}^{\infty} C_n$ cannot be the whole plane.

Roy O. Davies' theorem (1963)

Again, given an L_n-free set C_n with well-defined area for each $n = 1, 2, 3, \ldots$, the union $\bigcup_{n=1}^{\infty} C_n$ cannot be the whole plane. And yet:

Theorem

Given non-parallel lines $L_1, L_2, L_3, \ldots$, there are L_n-free sets C_n for $n = 1, 2, 3, \ldots$ that cover the plane, i.e., such that $\bigcup_{n=1}^{\infty} C_n = \mathbb{R}^2$.

How can this be?

- Using the Axiom of Choice, we can "construct" sets that are too complicated to have a well-defined area. Any such "construction" necessarily depends on an uncountable sequence of arbitrary choices.
- Like a sequence of coin tosses, the kind of choice sequence needed to prove Davies' Theorem does not follow any deterministic rule that we can write down.

Ordinals

With ordinals, "What's next?" always has a unique answer.

- The first ordinal after ordinal α is $\alpha + 1 = \alpha \cup \{\alpha\}$.
- The first ordinal after all the ordinals in a set S is $\bigcup_{\alpha \in S}(\alpha + 1)$.

$$
\begin{aligned}
0 &= \varnothing = \{\} \\
1 &= 0 \cup \{0\} = \{0\} \\
2 &= 1 \cup \{1\} = \{0, 1\} \\
3 &= 2 \cup \{2\} = \{0, 1, 2\} \\
n + 1 &= n \cup \{n\} = \{0, \dots, n-1, n\} \\
\omega &= \bigcup \{n + 1 \mid n \text{ finite}\} = \{0, 1, 2, \dots\} \\
\omega + 1 &= \omega \cup \{\omega\} = \{0, 1, 2, \dots, \omega\} \\
\omega + 2 &= \omega + 1 \cup \{\omega + 1\} = \{0, 1, 2, \dots, \omega, \omega + 1\} \\
\omega + \omega &= \bigcup \{\omega + n + 1 \mid n \text{ finite}\} = \{0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots\} \\
\alpha &= \{\beta \mid \beta < \alpha\}
\end{aligned}
$$

Cardinals

- A *cardinal* is an ordinal α such that there is no **bijection** to α from any $\beta < \alpha$.
- $0, 1, 2, 3, \ldots, n, \ldots, \omega$ are all cardinals.
- None of $\omega + 1, \omega + 2, \ldots, \omega + n, \ldots, \omega + \omega$ are cardinals.
- For example, there is a bijection from ω to $\omega + \omega$:

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & 2n & 2n+1 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\ \omega & 0 & \omega & \omega+1 & \cdots & n & \omega+n & \cdots \end{array}$$

- If there is a bijection to a set A from at least one ordinal, then define the *cardinality* $|A|$ to be the first such ordinal.
- If α is an ordinal, then $|\alpha|$ exists and $|\alpha| \leq \alpha$.
- If α is an cardinal, then $|\alpha| = \alpha$.
- If $|A|$ exists, then $|A|$ is a cardinal.

Choices

A set A is called *countable* if $|A| \leq \omega$, *i.e.*, if there is a bijection to A from some ordinal $\alpha \leq \omega$.

Theorem (Cantor, 1874)

The real line $\mathbb{R}$ is uncountable. (And, hence, so is the plane $\mathbb{R}^2$.)

Cantor proved his theorem without the Axiom of Choice.

Theorem (Zermelo, 1904)

Given any set A (which could be $\mathbb{R}$ or $\mathbb{R}^2$), there is a bijection F from some ordinal to A.

By Zermelo's theorem, every set A has a cardinality $|A|$.

Though many earlier proofs had used infinite sequences of choices, Zermelo's proof was the first to explicitly formulate the Choice as an axiom.

Proof outline for Zermelo's theorem

- For every proper subset $S \subsetneq A$, **choose** a point $c(S) \in A \setminus S$.
- Call a map G from an ordinal α into A "obedient" if
$$G(\beta) = c(\{G(\gamma) \mid \gamma < \beta\}) \text{ for all } \beta < \alpha.$$
- For any two obedient maps G and H, one of them extends the other. Why? If β is the first ordinal where $G(\beta) \neq H(\beta)$, then $G(\beta)$ and $H(\beta)$ both equal $c(\{G(\gamma) \mid \gamma < \beta\})$. Contradiction!
- Therefore, the union of all obedient maps is a maximal obedient map.
- The range R of the maximal obedient map $F\colon \alpha \to A$ has range A because otherwise $F(\alpha) = c(R)$ would extend F to $\alpha + 1$. $\square$

Defining Davies' tree

- By a *tree* we mean a set with an irreflexive acyclic binary relation "is a child of" and with a unique element called the *root* that is not a child of anything.
- Tree elements are called *nodes*.
- Each node of *Davies' tree* is a finite sequence of ordinals $a = (\alpha_1, \ldots, \alpha_n)$.
- The root node is (α_1) where $\alpha_1 = |\mathbb{R}^2|$.
- If $a = (\alpha_1, \ldots, \alpha_n)$ is a node and α_n is countable, then a has no children.
- A childless node is called a *leaf* of the tree.
- If $a = (\alpha_1, \ldots, \alpha_n)$ is a node and α_n is uncountable, then its children are all the nodes of the form $(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ where $\alpha_{n+1} < |\alpha_n|$.
- Note that $\alpha_{n+1} < |\alpha_n|$ implies $\alpha_{n+1} < \alpha_n$

Well ordering the leaves

- Order the leaves of Davies' tree lexicographically: $(\beta_1, \ldots, \beta_m) \triangleleft (\alpha_1, \ldots, \alpha_n)$ if $\alpha_k < \beta_k$ at the first k where $\alpha_k \neq \beta_k$.
- The lexicographic ordering $\triangleleft$ of the leaves of Davies' tree is a *well ordering*, meaning that there is bijection h from some ordinal δ to the set of all leaves such that $\alpha < \beta$ implies $h(\alpha) \triangleleft h(\beta)$.
- Therefore, for every set S of leaves, if there is some leaf after every leaf in S, then there is a first leaf after every leaf in S.
- The proof that $\triangleleft$ is a well-ordering uses the fact that Davies' tree has no infinite chain of descendants (α_1), (α_1, α_2), $(\alpha_1, \alpha_2, \alpha_3)$, ... because there is no infinite decreasing sequence of ordinals $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$
- The construction of h is like the proof of Zermelo's theorem, except that the Axiom of Choice is not used. (h is unique!)

Proof outline of well ordering of leaves

- A set S of leaves is an *initial segment* if $b \lhd a \in S$ implies $b \in S$.
- Let H be the set of all order isomorphisms from ordinals to initial segments of the set of leaves.
- For any two $f, g \in H$, one extends the other. Why? Because if α were the first ordinal where $f(\alpha) \neq g(\alpha)$, then $f(\alpha)$ and $g(\alpha)$ would both be the $\lhd$-least leaf after the leaves $f(\beta) = g(\beta)$ for $\beta < \alpha$. Contradiction!
- Therefore, the union $h = \bigcup H$ is a maximal element of H.
- If the range of h did not include all leaves, then we could choose α_1, then α_2, then α_3, and so on, with each α_k chosen *least possible* such that $(\alpha_1, \ldots, \alpha_k)$ has a descendant not in the range of h.
- Eventually we would obtain a leaf $a = (\alpha_1, \ldots, \alpha_n)$ lexicographically least among leaves not in the range of h.
- But then $h(\delta) = a$ would extend h to $\delta + 1$. Contradiction! $\square$

Closures

- Given a set A and a binary function g, define the *g-closure* of A to be $\bigcup_{n<\omega} A_n$ where $A_0 = A$ and $A_{n+1} = A_n \cup g[A_n^2]$.
- For example, if $A = \{5, -3\}$ and $g = +$, then the g-closure of A is $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- If B is the g-closure of A, then $g[B^2] \subset B$.
- Given a set A and a countable set Γ of binary functions, define *Γ-closure* of A to be $\bigcup_{n<\omega} A_n$ where $A_0 = A$ and $A_{n+1} = A_n \cup \bigcup_{g\in\Gamma} g_m[A_n^2]$.
- If B is the Γ-closure of A, then B is *Γ-closed*, by which we mean that $g[B^2] \subset B$ for all $g \in \Gamma$.
- If A is countable, then the Γ-closure of A is countable.
- If A is uncountable, then the Γ-closure of A has the same cardinality as A.

Labeling Davies' tree

- Let $(L_n)_{n<\omega}$ be a sequence of non-parallel lines.
- For each pair of distinct lines (L_m, L_n) and each pair of points (p, q), let $g_{m,n}(p, q)$ be the unique point on both the line through p parallel to L_m and the line through q parallel to L_n.
- Let Γ be the set of all these $g_{m,n}$. This Γ is countable.
- We will use Choice to construct a "labelling" function D such that for each node a of Davies' tree, $D(a)$ is Γ-closed subset of the plane.

Labeling Davies' tree

Starting from the root, we will recursively choose a Γ-closed subset $D(a)$ of the plane for each node $a = (\alpha_1, \ldots, \alpha_n)$ of Davies' tree then $|D(a)| = |\alpha_n|$.

- Label the root (α_1) with $D(\alpha_1) = \mathbb{R}^2$; note that $|\alpha_1| = ||\mathbb{R}^2|| = |\mathbb{R}^2| = |D(\alpha_1)|$.
- Given a node $a = (\alpha_1, \ldots, \alpha_n)$ with α_n uncountable and with label $D(a)$ such that $|\alpha_n| = |D(a)|$, label the children of a:
- **Choose** a bijection f from $|\alpha_n|$ to $D(a)$.
- For each child $b = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ where $\alpha_{n+1} < |\alpha_n|$, let $D(b)$ be the Γ-closure of $\{f(\beta) \mid \beta < \alpha_{n+1}\}$.
- If α_{n+1} is uncountable, then $|D(b)| = |\alpha_{n+1}|$ because $\{f(\beta) \mid \beta < \alpha_{n+1}\}$ is an uncountable set of cardinality $|\alpha_{n+1}|$ and its Γ-closure has the same cardinality.

Essential properties of Davies' tree labels

- If a has children, then its label $D(a)$ is the union of all labels of children of a.
- Hence, $\mathbb{R}^2 = D(\alpha_1)$ is the union of all labels of leaves.
- Each leaf has a countable label.
- Each node's label is Γ-closed.
- Sibling nodes' labels increase: if $b = (\alpha_1, \ldots, \alpha_n, \beta)$, $c = (\alpha_1, \ldots, \alpha_n, \gamma)$, and $\beta < \gamma$, then $D(b) \subset D(c)$.
- Hence, any union of the form $\bigcup_{\beta<\gamma} D(\alpha_1, \ldots, \alpha_n, \beta)$ is Γ-closed.

Fundamental Lemma. For each leaf $a = (\alpha_1, \ldots, \alpha_n)$, the union $\bigcup_{b \lhd a} D(b)$ of all labels of lexicographically earlier leaves $b \lhd a$ is the union of a **finite** list of Γ-closed sets: $\bigcup_{\beta<\alpha_k} D(\alpha_1, \ldots, \alpha_{k-1}, \beta)$ for $k = 2, 3, 4, \ldots, n$.

Partitioning the plane into L_n-free sets

- We will construct a map $C\colon \mathbb{R}^2 \to \omega$ such that each $C^{-1}\{n\}$ is L_n-free.
- Construct C one leaf label at a time.
- Given a (possibly empty) proper initial segment S of the leaves and a map $C\colon \bigcup_{b\in S} D(b) \to \omega$ such that each $C^{-1}\{n\}$ is L_n-free, we will extend the domain of C to contain $D(a)$ where a is the first leaf after S.
- By the Fundamental Lemma, the domain of C is the union of a **finite list** $A_0, \ldots, A_{r-1}$ of Γ-closed sets.
- Since $D(a)$ is countable, we may **choose** a bijection p from some $\lambda \leq \omega$ to the set of "new points" $D(a) \setminus \mathrm{dom}(C)$.
- Extend the domain of C to these new points one at a time...

Partitioning a new leaf label

- Again, the set of "old points" already in the domain of C is a finite union of Γ-closed sets $A_0, \ldots, A_{r-1}$.
- Again, $\lambda \leq \omega$ and the "new points" are $p(k)$ for $k < \lambda$.
- Given $k < \lambda$ and $C(p(j))$ for $j < k$, define $C(p(k))$ as follows.
- Given any pair of distinct lines (L_m, L_n), any of the sets A_s, and any pair of old points $u, v \in A_s$, the line through u parallel to L_m and the line through v parallel to L_n do not intersect at the new point $p(k)$ because A_s is Γ-closed.
- Therefore, for each $s < r$, at most one "bad" $m < \omega$ is such that the line through $p(k)$ parallel to L_m intersects A_s.
- Also, for each $j < k$, at most one "bad" $m < \omega$ is such that the line through $p(j)$ and $p(k)$ is parallel to L_m.
- Let $C(p(k))$ be the least $n < \omega$ not in the finite set of bad m's above. This keeps $C^{-1}\{n\}$ L_n-free. $\square$

Davies trees in the 21st century

- Davies' 1963 proof used a kind of labeled tree not seen in print again until in Jackson and Mauldin's 2002 proof that there is a set $S \subset \mathbb{R}$ that intersects every isometric copy of $\mathbb{Z}^2$ at exactly one point.
- I learned about Davies' technique from Arnold Miller in a 2005 graduate course.
- In 2006, I used it to prove a theorem that became part of a 2008 publication about set-theoretic topology after modifying Davies' technique into something simpler for experts in set theory to use.
- I have since applied this modified technique in other publications concerning Boolean algebras.

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