On the order theory of local bases

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Order theory Preliminaries

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- A preorder P is κ-**directed** if every subset smaller than κ has an (upper) bound in P.
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Convention

Order sets like $[\lambda]^\kappa$ and $2^{<\kappa}$ by $\subseteq$.

Classifying preorders

Definition

Two preorders P and Q are **mutually cofinal** if they are isomorphic to cofinal suborders of a common third preorder R.

Lemma (M., 2005)

If P and Q are mutually cofinal and P is almost κ-short, then Q is almost κ-short.

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Definition (Tukey, 1940)

- P is **Tukey-below** Q, or $P \leq_T Q$, if there exists $f\colon P \leq_T Q$, *i.e.*, $f\colon P \to Q$ sends unbounded sets to unbounded sets.
- $P \equiv_T Q$ means $P \leq_T Q \leq_T P$.

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Tukey types aren't cofinal types...

$2^{<\omega_1} \equiv_T [\mathfrak{c}]^1$ and $[\mathfrak{c}]^1$ is flat, but $2^{<\omega_1}$ is not almost flat.

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Definition

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Theorem (M., 2007)

If P is directed and $\mathrm{cf}(P) \geq \aleph_0$, then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3):

1. $[\mathrm{cf}(P)]^{<\kappa} \leq_T P$.
2. P has a $(\mathrm{cf}(P), \kappa)$-blossom.
3. P is almost κ-short.

If also $\kappa = \mathrm{cf}(\kappa)$ and $|[\mathrm{cf}(P)]^{<\kappa}| = \mathrm{cf}(P)$, then (1) $\Leftarrow$ (2).

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Proof

- (1) $\Rightarrow$ (2): If $f\colon [\mathrm{cf}(P)]^{<\kappa} \leq_T P$, then $\langle f(\{\alpha\})\rangle_{\alpha<\mathrm{cf}(P)}$ is a $(\mathrm{cf}(P), \kappa)$-blossom.

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- (2) $\Rightarrow$ (3): Given a $(\mathrm{cf}(P), \kappa)$-blossom b and $c\colon \mathrm{cf}(P) \to P$ with cofinal range, let $d(\alpha) \geq b(\alpha), c(\alpha)$ for all α; $\mathrm{ran}(d)$ is cofinal and κ-short.

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- (2) $\Rightarrow$ (3): Given a $(\mathrm{cf}(P), \kappa)$-blossom b and $c\colon \mathrm{cf}(P) \to P$ with cofinal range, let $d(\alpha) \geq b(\alpha), c(\alpha)$ for all α; $\mathrm{ran}(d)$ is cofinal and κ-short.
- (2) $\Rightarrow$ (1): Given a $(\mathrm{cf}(P), \kappa)$-blossom b and an injective $h\colon [\mathrm{cf}(P)]^{<\kappa} \to \mathrm{cf}(P)$, we have $b \circ h\colon [\mathrm{cf}(P)]^{<\kappa} \leq_T P$.

Topological preliminaries

Convention

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- $\tau(X)$ is the set of open subsets of X.
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Definition

- A **local base** at p is a cofinal subset of $\tau(p, X)$.
- A π-**base** is a cofinal subset of $\tau^+(X)$.
- A **base** is a subset $\mathcal{B}$ of $\tau(X)$ that includes a local base at every point.

The **weight** $w(X)$ of X is the least $\kappa \geq \aleph_0$ such that X has a base that is of size $\leq \kappa$.	The **Noetherian type** $\mathrm{Nt}(X)$ of X is the least $\kappa \geq \aleph_0$ such that X has a base that is κ-short.
The π-**weight** $\pi(X)$ of X is the least $\kappa \geq \aleph_0$ such that X has a π-base that is of size $\leq \kappa$.	The **Noetherian** π-**type** $\pi\mathrm{Nt}(X)$ of X is the least $\kappa \geq \aleph_0$ such that X has a π-base that is κ-short.
The **character** $\chi(p, X)$ of p in X is the least $\kappa \geq \aleph_0$ such that p has a local base that is of size $\leq \kappa$.	The **local Noetherian type** $\chi\mathrm{Nt}(p, X)$ of p in X is the least $\kappa \geq \aleph_0$ such that p has a local base that is κ-short.
$\chi(X) = \sup_{p\in X} \chi(p, X)$	$\chi\mathrm{Nt}(X) = \sup_{p\in X} \chi\mathrm{Nt}(p, X)$

History

- Malykhin, Peregudov, and Šapirovskiĭ studied the properties $\mathrm{Nt}(X) \leq \aleph_1$, $\pi\mathrm{Nt}(X) \leq \aleph_1$, $\mathrm{Nt}(X) = \aleph_0$, and $\pi\mathrm{Nt}(X) = \aleph_0$ in the 1970s and 1980s.
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Order-theoretic definitions

$\pi(X)$ is the least $\kappa \geq \aleph_0$ such that $\mathrm{cf}(\tau^+(X)) \leq \kappa$.	$\pi\mathrm{Nt}(X)$ is the least $\kappa \geq \aleph_0$ such that $\tau^+(X)$ is almost κ-short.
$\chi(p, X)$ is the least $\kappa \geq \aleph_0$ such that $\mathrm{cf}(\tau(p, X)) \leq \kappa$.	$\chi\mathrm{Nt}(p, X)$ is the least $\kappa \geq \aleph_0$ such that $\tau(p, X)$ is almost κ-short.

Easy upper bounds

Lemma

Every preorder P is almost $\mathrm{cf}(P)$-short.

Corollary

For all spaces X,

- $\chi\mathrm{Nt}\,(p, X) \leq \chi(p, X)$;
- $\chi\mathrm{Nt}\,(X) \leq \chi(X)$;
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Even easier:

Every P is $|P|^+$-short, so $\mathrm{Nt}\,(X) \leq w(X)^+$.

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Example

$\mathrm{Nt}\,(\beta\mathbb{N}) = w(\beta\mathbb{N})^+ = \mathfrak{c}^+$ because $\pi(\beta\mathbb{N}) = \aleph_0 < \mathrm{cf}(w(\beta\mathbb{N}))$.

Passing to subsets

Applying mutual cofinality

- If $\mathcal{B}$ is a π-base of X, then $\mathcal{B}$ includes a $\pi\mathrm{Nt}(X)$-short π-base of X.
- If $\mathcal{B}$ is a local base at p in X, then $\mathcal{B}$ includes a $\chi\mathrm{Nt}(X)$-short local base at p in X.

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Theorem (M., 2007)

Every metrizable space has a flat base.

Proof: For each $n < \omega$, pick a locally finite open cover refining the balls of radius 2^{-n}. Take the union.

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Example (M., 2009)

Set $X = \mathbb{Z}^{\omega}$. Let $\mathcal{B}$ be the set of all sets of the form $U_{s,n}$ where $s \in \mathbb{Z}^{<\omega}$, $n < \omega$, and $U_{s,n}$ is the set of all $f \in X$ such that $s^\frown i \subseteq f$ for some $i \in [-n, n]$. $\mathcal{B}$ a base of X, but $\mathcal{B}$ has no flat subcover.

Blossoms and splitters

Applying directedness

If $p \in X$ is not isolated, then $\chi\mathrm{Nt}(p, X) \leq \kappa$ if and only if $\tau(p, X)$ has a $(\chi(p, X), \kappa)$-blossom, which is just a $\chi(p, X)$-sequence $\vec{U}$ of neighborhoods of p such that $p \notin \operatorname{int} \bigcap_{\alpha \in I} U_\alpha$ for all $I \in [\chi(p, X)]^\kappa$.

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Definition

A (λ, κ)-**splitter** of X is a λ-sequence $\vec{\mathcal{F}}$ of finite open covers of X such that $\operatorname{int} \bigcap_{\alpha \in I} U_\alpha = \varnothing$ for all $I \in [\chi(p, X)]^\kappa$ and $\vec{U} \in \prod_{\alpha \in I} \mathcal{F}_\alpha$.

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Lemma

If X has a $(w(X), \kappa)$-splitter, then $\mathrm{Nt}\,(X) \leq \kappa$.

Question (M., 2007)

Does $\mathrm{Nt}\,(\beta\omega \setminus \omega) \leq \kappa$ imply $\beta\omega \setminus \omega$ has a $(\mathfrak{c}, \kappa)$-splitter in ZFC? (There can be no counterexamples if $\mathfrak{c}$ is regular.)

Easy applications of blossoms and splitters

Theorem

If $X = \prod_{\alpha<\kappa} X_\alpha$ and $|X_\alpha| > 1$ for all $\alpha < \kappa$, then

- $\kappa \geq \chi(p, X) \Rightarrow \chi\mathrm{Nt}\,(p, X) = \aleph_0$;
- $\kappa \geq \chi(X) \Rightarrow \chi\mathrm{Nt}\,(X) = \aleph_0$;
- $\kappa \geq \pi(X) \Rightarrow \pi\mathrm{Nt}\,(X) = \aleph_0$;
- $\kappa \geq w(X) \Rightarrow \mathrm{Nt}\,(X) = \aleph_0$.

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- $\kappa \geq \pi(X) \Rightarrow \pi\mathrm{Nt}\,(X) = \aleph_0$;
- $\kappa \geq w(X) \Rightarrow \mathrm{Nt}\,(X) = \aleph_0$.

Proof (essentially (Malykhin, 1981))

First claim: For each $\alpha < \chi(p, X)$, choose a nontrivial open neighborhood U_α of $p(\alpha)$. Since all open boxes in the product topology have finite support, $\langle \pi_\alpha^{-1}[U_\alpha] \rangle_{\alpha<\kappa}$ is a $(\chi(p, X), \aleph_0)$-blossom for $\tau(p, X)$.

Corollary

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- $\mathrm{Nt}\left(X^{w(X)}\right) = \aleph_0$.
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Passing to subsets again

Definition

A space X is **homogeneous** if for all $p, q \in X$, there is a bijection $f : X \to X$ with $f(p) = q$ and f and f^{-1} continuous.

Theorem (M., 2009)

Let $\mathcal{B}$ be a base of X. $\mathcal{B}$ includes an $\mathrm{Nt}(X)$-short base of X if

- X is metrizable and X is locally compact or σ-compact,
- X is compact and $\chi(p, X) = w(X)$ for all $p \in X$, or
- X is compact, homogeneous, and $w(X)$ is regular or strong limit.

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About the proof

- For the second case, we build a $(w(X), \kappa)$-splitter consisting of subcovers of an arbitrary base.
- For the third case, we use Misčenko's Lemma to deduce that the second case holds or $\mathrm{Nt}(X) = w(X)^+$.

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- It follows that every known CHS has cellularity at most $\mathfrak{c}$. (Why? Easy: $\mathfrak{c}^+$ is a caliber of any such space.)
- Van Douwen's Problem asks whether $c(X) \leq \mathfrak{c}$ for every CHS X. **This is open after $\sim$40 years, in all models of ZFC.**

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- Van Douwen's Problem asks whether $c(X) \leq \mathfrak{c}$ for every CHS X. **This is open after $\sim$40 years, in all models of ZFC.**
- It also follows that every known CHS has Noetherian type at most $\mathfrak{c}^+$. (Why? Not as easy...)

Sharp bounds

Example (Maurice, 1964)

The lexicographically ordered space $X = 2^{\omega \cdot \omega}_{\mathrm{lex}}$ is a CHS satisfying $c(X) = \mathfrak{c}$.

Example (Peregudov, 1997)

The double-arrow space X is compact, homogeneous, and $\mathrm{Nt}(X) = \mathfrak{c}^{+}$.

Light factors

Theorem (M., 2007)

If X is CHS and a continuous image of a product of compacta all with weight at most λ, then $\mathrm{Nt}(X) \leq \lambda^+$. If also $\lambda = \aleph_0$ (*i.e.*, X is **dyadic**), then $\mathrm{Nt}(X) = \aleph_0$.

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Some ideas from the proof

- A **long** κ**-approximation sequence** (for regular κ) is an $\in$-chain $\vec{M}$ of elementary substructures of $H(\theta)$ with $|M_\alpha| \subseteq \kappa \cap M_\alpha \in \kappa \in M_\alpha$ and $\vec{M} \upharpoonright \alpha \in M_\alpha$.

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- (A. Miller) Generalizing (Jackson, Mauldin, 2002), given $\vec{M}$ as above, there exists $\vec{\Sigma}$ such that $\Sigma_\alpha \in [M_\alpha]^{<\aleph_0}$, $\bigcup \Sigma_\alpha = \bigcup(\vec{M} \upharpoonright \alpha)$, and $N \prec H_\theta$ for all $N \in \Sigma_\alpha$.

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- (A. Miller) Generalizing (Jackson, Mauldin, 2002), given $\vec{M}$ as above, there exists $\vec{\Sigma}$ such that $\Sigma_\alpha \in [M_\alpha]^{<\aleph_0}$, $\bigcup \Sigma_\alpha = \bigcup(\vec{M} \upharpoonright \alpha)$, and $N \prec H_\theta$ for all $N \in \Sigma_\alpha$.
- The quotient maps $\pi \colon X \to X/M_\alpha$ are **open**.

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If X is CHS and a continuous image of a product of compacta all with weight at most λ, then $\mathrm{Nt}(X) \leq \lambda^+$. If also $\lambda = \aleph_0$ (*i.e.*, X is **dyadic**), then $\mathrm{Nt}(X) = \aleph_0$.

Some ideas from the proof

- A **long κ-approximation sequence** (for regular κ) is an $\in$-chain $\vec{M}$ of elementary substructures of $H(\theta)$ with $|M_\alpha| \subseteq \kappa \cap M_\alpha \in \kappa \in M_\alpha$ and $\vec{M} \upharpoonright \alpha \in M_\alpha$.
- (A. Miller) Generalizing (Jackson, Mauldin, 2002), given $\vec{M}$ as above, there exists $\vec{\Sigma}$ such that $\Sigma_\alpha \in [M_\alpha]^{<\aleph_0}$, $\bigcup \Sigma_\alpha = \bigcup(\vec{M} \upharpoonright \alpha)$, and $N \prec H_\theta$ for all $N \in \Sigma_\alpha$.
- The quotient maps $\pi\colon X \to X/M_\alpha$ are **open**.
- We can build a κ-short base of X by taking the union of pullbacks of well-chosen bases of these quotients.

π-character is crucial.

Definition

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- If $\pi\chi(p, X) < \operatorname{cf} \kappa = \kappa \leq \chi(p, X)$ for some $p \in X$, then $\mathrm{Nt}(X) > \kappa$. (Essentially (Peregudov, 1997))

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- If $h\colon \prod_{i \in I} X_i \to X$ is a continuous surjection, $\pi\chi(p, X) = w(X)$ for all $p \in X$, X_i is compact, and $w(X_i) \leq \lambda$ (for all $i \in I$), then $\mathrm{Nt}\,(X) \leq \lambda$.

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About the proof

This time, we don't need long κ-approximation sequences. Continuous elementary chains work just fine.

Does every CHS have a flat local base?

Another Pattern

Every known CHS X satisfies $\pi\mathrm{Nt}\,(X) \leq \aleph_1$ and $\chi\mathrm{Nt}\,(X) = \aleph_0$.

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Attacking Van Douwen's Problem

- If we found a model of GCH with a CHS X with a local base $\mathcal{B}$ such that $\mathcal{B}$ is not almost $\aleph_1$-short, then $c(X) > \mathfrak{c}$.
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- (Arhangel′skiĭ, 2005) If a product of linear orders is a CHS, then all factors are first countable, and hence have weight at most $\mathfrak{c}$.

π-character is crucial, again.

Assuming GCH, $\chi\mathrm{Nt}(X) \leq c(X)$ if X is a CHS.

Proof

- **Lemma** (M. 2007). If X is compact and $\pi\chi(p, X) \geq \kappa$ for all $p \in X$, then $\tau(q, X)$ has $(\kappa, \aleph_0)$-blossom for some $q \in X$.

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- Therefore, given a CHS X, we have $\chi\mathrm{Nt}(X) = \aleph_0$ or $\pi\chi(X) < \chi(X)$.

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- (Arhangel′skiĭ, Ismail) If Y is a CHS, then $2^{\chi(Y)} \leq |Y| \leq 2^{\pi\chi(Y)c(Y)}$.

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- So, assuming GCH, $\pi\chi(X) < \chi(X)$ implies $\chi\mathrm{Nt}(X) \leq \chi(X) \leq c(X)$.

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- (M., Spadaro, 2010) If X is compact and $\pi\chi(p, X) < w(X)$ for a dense set of points, then $\mathrm{Nt}\,(X) \geq w(X)$, and $\mathrm{Nt}\,(X) = w(X)^+$ if $w(X)$ is regular.

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Examples

- (M., 2010) If $X = D_{\aleph_\omega} \cup \{\infty\}$, then $\pi\chi(X) = \aleph_0$, $w(X) = \aleph_\omega$, and $\mathrm{Nt}(X) = \aleph_{\omega+1}$.
- (M., 2010) If $X = \prod_{n<\omega}(D_{\aleph_n} \cup \{\infty\})$, then $\pi\chi(X) = \aleph_0$, $w(X) = \aleph_\omega$, and $\mathrm{Nt}(X) = \aleph_\omega$.

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Definition (Van Douwen)

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- It is also unknown whether every PHC X has a flat local base.
- Perhaps an easier question: Does GCH imply $\chi\mathrm{Nt}(X) \leq c(X)$ for all PHC X?

A partial answer

Definition

$d(X)$ is the least $\kappa \geq \aleph_0$ such that some $D \in [X]^{\leq\kappa}$ is dense in X.

Perhaps an even easier question:

Does GCH imply $\chi\mathrm{Nt}(X) \leq d(X)$ for all PHC X?

Theorem (M., Ridderbos, 2007)

Given GCH, X PHC, and $\max_{p\in X}\chi(p, X) = \mathrm{cf}(\chi(X)) > d(X)$, there is a nonempty open $U \subseteq X$ such that $\chi\mathrm{Nt}(p, X) = \aleph_0$ for all $p \in U$.

If we stop worrying about homogeneity. . .

Sometimes compactness doesn't matter.

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Product spaces can surprise you.

- (Todorčević, 1985) If $\mathrm{cf}(\kappa) = \kappa = \kappa^{\aleph_0}$, then there exist directed P, Q with $P, Q <_T P \times Q \equiv_T [\kappa]^{<\aleph_0}$.
- (M., 2010) Using these P and Q, we can build X, Y such that $\chi\mathrm{Nt}(X) = \chi\mathrm{Nt}(Y) = \aleph_1$ and $\chi\mathrm{Nt}(X \times Y) = \aleph_0$.

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- (Sparado, 2010) X, Y can be modified to get Z, W such that $\mathrm{Nt}\,(Z) = \mathrm{Nt}\,(W) = \aleph_1$ and $\mathrm{Nt}\,(Z \times W) = \aleph_0$.

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- (Spadaro, 2008) There are compact K, L with $\mathrm{Nt}\,(K) = \aleph_2$, $\mathrm{Nt}\,(L) = \aleph_3$, and $\mathrm{Nt}\,(K \times L) = \aleph_1$.

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- Open: Is $\mathrm{Nt}\,(X^2) \neq \mathrm{Nt}\,(X)$ possible?

Powers

- (M., 2010) We can also use the previous P and Q to build an example of $\chi\mathrm{Nt}\left(\langle p, q\rangle, X^2\right) = \aleph_0 < \aleph_1 = \chi\mathrm{Nt}\,(p, X) = \chi\mathrm{Nt}\,(q, X)$.

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- (M., 2010) We can also use the previous P and Q to build an example of $\chi\mathrm{Nt}\left(\langle p,q\rangle, X^2\right) = \aleph_0 < \aleph_1 = \chi\mathrm{Nt}\,(p,X) = \chi\mathrm{Nt}\,(q,X)$.
- (M., 2007) If $f\colon X \to Y$ is continuous and open at p, then $\chi\mathrm{Nt}\,(p,X) \leq \chi\mathrm{Nt}\,(f(p),Y)$ (and $\tau(p,X) \geq_T \tau(f(p),Y)$).
- Hence, $0 < \alpha < \beta \Rightarrow \chi\mathrm{Nt}\left(p, X^\beta\right) \leq \chi\mathrm{Nt}\,(p \restriction \alpha, X^\alpha)$.
- (M., 2009) If $0 < \gamma < \omega_1$, then $\chi\mathrm{Nt}\,(p^\gamma, X^\gamma) = \chi\mathrm{Nt}\,(p,X)$ and $\chi\mathrm{Nt}\,(X^\gamma) = \chi\mathrm{Nt}\,(X)$.
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- (M., 2005) If $\chi(p, X) \leq \gamma$ and $|X| > 1$, then $\chi\mathrm{Nt}\,(p^\gamma, X^\gamma) = \aleph_0$.

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Definition

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Example (M., 2009)

- If $p \in X = \prod_{\alpha<\lambda}^{(\lambda)} 2^\alpha$ and λ is strongly inaccessible, then $\mathrm{split}_\mu(p, X) = \aleph_0$ for all $\mu < \lambda$, but $\mathrm{split}_\lambda(p, X) = \chi\mathrm{Nt}\,(p, X) = \lambda$.
- The proof's essential ingredient runs short an elementary-submodel proof of the Erdös-Rado Theorem.

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Example (M., 2009)

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- The key lemma for the proof is that the set of countably supported maps from ω_1 to ω (with the product ordering) does not have an $(\omega_1, \aleph_0)$-blossom.
- Why? If $F\colon \omega_1 \to \mathrm{Fn}(\omega_1, \omega, \aleph_1)$, $F \in M \prec H(\aleph_2)$, and $|M| = \aleph_0$, then we can use reflect properties of $F(\omega_1 \cap M)$ to find infinitely many $F(\alpha) \in M$ all dominated by a single $g \in \mathrm{Fn}(\omega_1, \omega, \aleph_1)$.

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- Let $p \in X = \prod_{\alpha<\aleph_\omega}^{(\aleph_1)} 2$. We then have $\chi(p, X) = \pi(X) = w(X) = \aleph_\omega^{\aleph_0}$.

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- If $\square_{\aleph_\omega}$ and $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}$, then $\mathrm{Nt}\,(X) = \pi\mathrm{Nt}\,(X) = \chi\mathrm{Nt}\,(p, X) = \aleph_1$. (Why? We can use Bernstein sets and a locally countable $S \subseteq [\aleph_\omega]^{\aleph_0}$ of size $\aleph_{\omega+1}$ to build an $\aleph_1$-short base...)

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- (Soukup) $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0) \Rightarrow \mathrm{Nt}(X) \geq \chi\mathrm{Nt}(p, X) \geq \aleph_2$. (The hypothesis is consistent relative (roughly) to a huge cardinal (Levinski, Magidor, Shelah, 1990).)

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- Open: Can we have $\pi\mathrm{Nt}\,(X) > \aleph_1$? Equivalently, can $\langle \mathrm{Fn}(\aleph_\omega, 2, \aleph_1), \subseteq \rangle$ fail to be almost $\aleph_1$-short?

Noetherian spectra

Another application of Bernstein sets (M., 2009)

If $\mathfrak{c} \geq \kappa$ and κ is weakly inacessible, then there is a Lindelöf linear order with Noetherian type κ..

Excluded Noetherian types (M., 2008)

- The compact linear orders attain all Noetherian types except $\aleph_1$ and weak inaccessibles.

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- The dyadic compacta do not attain Noetherian type $\aleph_1$.
- Open: do the dyadic compacta attain weakly inaccessible Noetherian types?
- Open: do the dyadic compacta attain Noetherian type $\aleph_{\omega+1}$?

Local bases in $\beta\omega \setminus \omega$

Convention

- If $\mathcal{U}$ is an ultrafilter on ω, then order $\mathcal{U}$ by $\supseteq$.
- Let $\mathcal{U}_*$ denote $\mathcal{U}$ ordered by $\supseteq^*$ (containment modulo $[\omega]^{<\aleph_0}$).

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Isbell's Problem

ZFC proves there exists $\mathcal{U} \in \beta\omega \setminus \omega$ such that $\mathcal{U}_* \equiv_T \mathcal{U} \equiv_T [\mathfrak{c}]^{<\aleph_0}$.
Does ZFC prove there exists $\mathcal{V} \in \beta\omega \setminus \omega$ such that $\mathcal{V} \not\equiv_T [\mathfrak{c}]^{<\aleph_0}$?

$\mathcal{U}$ versus $\mathcal{U}_*$

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- If $\mathcal{U}_*$ is not $\aleph_1$-directed, then $\mathcal{V} \leq_T \mathcal{U}_*$ for some $\mathcal{V} \in \beta\omega \setminus \omega$.

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- If P is $\aleph_1$-directed and $\kappa \geq \aleph_0$, then $P \not\geq_T [\kappa]^{<\aleph_0}$.
- Hence, Isbell's Problem is equivalent to asking if ZFC proves there exists $\mathcal{U} \in \beta\omega \setminus \omega$ such that $\mathcal{U}_* \not\equiv_T [\mathfrak{c}]^{<\aleph_0}$.

Ultrafilter Tukey classes for $\supseteq^*$

- (M., 2008) Assuming $\mathfrak{p} = \mathfrak{c}$, for every regular $\kappa \in [\aleph_0, \mathfrak{c}]$, there exists $\mathcal{U}_* \equiv_T [\mathfrak{c}]^{<\kappa}$, which implies $\chi\mathrm{Nt}(\mathcal{U}, \beta\omega \setminus \omega) = \kappa$.

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- (Aviles, Todorčević, 2010) If $n \geq 2$, $\kappa < \mathfrak{m}_{\sigma-n-\mathrm{linked}}$, and $A_0, \ldots, A_n \subseteq \beta\omega \setminus \omega$ are disjoint open F_κ sets, then there are clopen $B_0 \supseteq A_0, \ldots, B_n \supseteq A_n$ such that $\bigcap_{i \leq n} B_i = \varnothing$.
- (M., 2010) It follows that $\mathcal{U}_* \not\equiv_T \kappa \times P$ for all $\mathcal{U} \in \beta\omega \setminus \omega$ if $\omega \leq \mathrm{cf}(\kappa) = \kappa < \sup_{n<\omega} \mathfrak{m}_{\sigma-n-\mathrm{linked}}$ and P is the union of at most κ-many κ^+-directed sets. *E.g.*, $\mathcal{U}_* \not\equiv_T \omega \times \omega_1$, and $\mathrm{MA}_{\aleph_1} \Rightarrow \mathcal{U}_* \not\equiv_T \omega_1 \times \omega_2$.

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- (M., 2009) Assuming $\mathfrak{t} = \mathfrak{c}$ and $\diamondsuit(S^{\mathfrak{c}}_{\omega})$ (which are implied by $\mathrm{MA} \wedge \mathfrak{c} = \aleph_2$), there exists $\mathcal{W}\colon 2^{\mathfrak{c}} \to \beta\omega \setminus \omega$ such that $\mathcal{W}(f)_* \not\leq_T \mathcal{W}(g)$ for all $f \neq g$.

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- Open: Does CH imply there exist $\mathcal{U}, \mathcal{V} \in \beta\omega \setminus \omega$ such that $\mathcal{U}_* \not\leq_T \mathcal{V}_* \not\leq_T \mathcal{U}_*$?

The (local) Noetherian (π-)type of $\beta\omega \setminus \omega$

ZFC proves each of the following statements.

- $\pi\mathrm{Nt}(\beta\omega \setminus \omega) = \mathfrak{h} \leq \mathfrak{s} \leq \mathrm{Nt}(\beta\omega \setminus \omega) \leq \mathfrak{c}^+$.
- $\chi\mathrm{Nt}(\beta\omega \setminus \omega) \leq \min\{\mathrm{Nt}(\beta\omega \setminus \omega), \mathfrak{c}\}$.
- *MA* $\Rightarrow \pi\mathrm{Nt}(\beta\omega \setminus \omega) = \mathfrak{c} \Rightarrow \mathrm{Nt}(\beta\omega \setminus \omega) = \mathfrak{c}$.
- $\mathfrak{r} = \mathfrak{c} \Rightarrow \mathrm{Nt}(\beta\omega \setminus \omega) \leq \mathfrak{c}$.
- $\mathfrak{r} < \mathfrak{c} \Rightarrow \mathrm{Nt}(\beta\omega \setminus \omega) \geq \mathfrak{c}$.
- $\mathfrak{r} < \mathrm{cf}\,\mathfrak{c} \Rightarrow \mathrm{Nt}(\beta\omega \setminus \omega) = \mathfrak{c}^+$.

Each of the following statements is consistent with ZFC.

- $\omega_1 = \pi\mathrm{Nt}(\beta\omega \setminus \omega) = \chi\mathrm{Nt}(\beta\omega \setminus \omega) = \mathrm{Nt}(\beta\omega \setminus \omega) < \mathfrak{c}$.
- $\omega_1 < \pi\mathrm{Nt}(\beta\omega \setminus \omega) = \chi\mathrm{Nt}(\beta\omega \setminus \omega) = \mathrm{Nt}(\beta\omega \setminus \omega) < \mathfrak{c}$.
- $\omega_1 = \pi\mathrm{Nt}(\beta\omega \setminus \omega) < \mathrm{Nt}(\beta\omega \setminus \omega) < \mathfrak{c}$.
- $\omega_1 < \pi\mathrm{Nt}(\beta\omega \setminus \omega) < \chi\mathrm{Nt}(\beta\omega \setminus \omega) = \mathfrak{c} < \mathrm{Nt}(\beta\omega \setminus \omega)$.