Higher-order amalgamation of algebraic structures

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Overlapping structures

Below are two overlapping group multiplication tables, $G_1 \cong C_2^2$ on the left and $G_2 \cong S_3$ on the right.

$\cdot$	z	xz	1	x	y	xy	y^2	xy^2
z	1	x	z	xz				
xz	x	1	xz	z				
1	z	xz	1	x	y	xy	y^2	xy^2
x	xz	z	x	1	xy	y	xy^2	y^2
y			y	xy^2	y^2	x	1	xy
xy			xy	y^2	xy^2	1	x	y
y^2			y^2	xy	1	xy^2	y	x
xy^2			xy^2	y	x	y^2	xy	1

In general, an indexed set $(A_i : i \in E)$ of algebraic structures is overlapping if, for all distinct $i, j \in E$, the algebraic operations of A_i and A_j, including any distinguished elements, agree when restricted to $A_i \cap A_j$.

An amalgamation

I say that G_1 and G_2 amalgamate as groups because there is a group $K \cong C_2 \times S_3$ containing G_1 and G_2 as subgroups:

$\cdot$	yz	xyz	y^2z	xy^2z	z	xz	1	x	y	xy	y^2	xy^2
yz	y^2	x	1	xy	y	xy^2	yz	xy^2z	y^2z	xz	z	xyz
xyz	xy^2	1	x	y	xy	y^2	xyz	y^2z	xy^2z	z	xz	yz
y^2z	1	xy^2	y	x	y^2	xy	y^2z	xyz	z	xy^2z	yz	xz
xy^2z	x	y^2	xy	1	xy^2	y	xy^2z	yz	xz	y^2z	xyz	z
z	y	xy	y^2	xy^2	$\mathbf{1}$	$\mathbf{x}$	$\mathbf{z}$	$\mathbf{xz}$	yz	xyz	y^2z	xy^2z
xz	xy	y	xy^2	y^2	$\mathbf{x}$	$\mathbf{1}$	$\mathbf{xz}$	$\mathbf{z}$	xyz	yz	xy^2z	y^2z
1	yz	xyz	y^2z	xy^2z	$\mathbf{z}$	$\mathbf{xz}$	$\mathbf{1}$	$\mathbf{x}$	$\mathbf{y}$	$\mathbf{xy}$	$\mathbf{y^2}$	$\mathbf{xy^2}$
x	xyz	yz	xy^2z	y^2z	$\mathbf{xz}$	$\mathbf{z}$	$\mathbf{x}$	$\mathbf{1}$	$\mathbf{xy}$	$\mathbf{y}$	$\mathbf{xy^2}$	$\mathbf{y^2}$
y	y^2z	xz	z	xyz	yz	xy^2z	$\mathbf{y}$	$\mathbf{xy^2}$	$\mathbf{y^2}$	$\mathbf{x}$	$\mathbf{1}$	$\mathbf{xy}$
xy	xy^2z	z	xz	yz	xyz	y^2z	$\mathbf{xy}$	$\mathbf{y^2}$	$\mathbf{xy^2}$	$\mathbf{1}$	$\mathbf{x}$	$\mathbf{y}$
y^2	z	xy^2z	yz	xz	y^2z	xyz	$\mathbf{y^2}$	$\mathbf{xy}$	$\mathbf{1}$	$\mathbf{xy^2}$	$\mathbf{y}$	$\mathbf{x}$
xy^2	xz	y^2z	xyz	z	xy^2z	yz	$\mathbf{xy^2}$	$\mathbf{y}$	$\mathbf{x}$	$\mathbf{y^2}$	$\mathbf{xy}$	$\mathbf{1}$

Amalgamation formalized

Given a category $\mathcal{C}$ consisting of a class of algebraic structures and all homomorphisms between them, and overlapping $(A_i : i \in E)$ in $\mathcal{C}$, we say that $A_1, \ldots, A_n$ amalgamate in $\mathcal{C}$ if there are embeddings $e_i \colon A_i \to B$ in $\mathcal{C}$ that commute with the identity embeddings $\mathrm{id} \colon A_i \cap A_j \to A_i$, $\mathrm{id}(x) = x$.

$$\begin{array}{ccc} A_i \cap A_j & \xrightarrow{\mathrm{id}} & A_i \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle e_i} \\ A_j & \xrightarrow{e_j} & B \end{array}$$

By embedding, I mean an injective homomorphism.
(Embeddings are always monomorphisms (if you know what those are), and the converse is true in most categories of interest. The category of divisible abelian groups is an exception.)

Schreier's Theorem

Call an indexed family of sets $(A_i : i \in E)$ a Δ-system or sunflower if there is root R such that $A_i \cap A_j = R$ for all distinct i, j.

Theorem (Schreier, 1927)

Every sunflower of groups $(G_i : i \in E)$ amalgamates in the category of groups. In particular, and two overlapping groups amalgamate as groups.

About the proof.

Choose isomorphisms $\psi_i \colon G_i \to H_i$ such that $H_i \cap H_j = \{1\}$. Let F be the free product consisting of all words $a_1 a_2 \cdots a_n$ where adjacent letters a_k, a_{k+1} are not in the same H_i. Let $K = F/N$ where N is the smallest normal subgroup of F containing all words of the form $\psi_i(r)\psi_j(r^{-1})$ where r is in the root.
Schreier proves that $\psi_i/N \colon G_i \to K$ is injective by means of a normal form lemma. □

Compactness I

Typically, if there are no finitary obstructions to amalgamation, then there are no obstructions at all.

Lemma

Suppose that:

- $A_i \cap A_j \in \mathcal{C}$ *for all* $i, j \in E$.
- *All diagrams in* $\mathcal{C}$ *have colimits.*
- $(A_i : i \in F)$ *amalgamates in* $\mathcal{C}$ *for all finite* $F \subset E$.

Then $(A_i : i \in E)$ *amalgamates in* $\mathcal{C}$.

Proof sketch.

For each finite $F \subset E$, let B_F and $\phi_{i,F} \colon A_i \to B_F$ for $i \in F$ be the colimit of the morphisms $\mathrm{id} \colon A_i \cap A_j \to A_i$ for $i, j \in F$. The morphisms $\phi_{i,F}$ must be injective. For $G \subset F \subset E$, there is a natural morphism $\chi_{F,G} \colon B_G \to B_F$, and it must also be injective. Let C and $\psi_F \colon B_F \to C$ for finite $F \subset E$ be the colimit of the morphisms $\chi_{F,G}$ for $G \subset F$. Then $\psi_{\{i\}} \colon A_i \to C$ must be injective for each $i \in E$. □

Compactness II

Lemma

Suppose that:

- $A_i \cap A_j \in \mathcal{C}$ *for all* $i, j \in E$.
- $\mathcal{C}$ *is axiomatized by a set of first-order formulas.*
- $(B_i : i \in E)$ *amalgamates in* $\mathcal{C}$ *for all finitely generated substructures* $B_i \subset A_i$ *with* $B_i \in \mathcal{C}$.

Then $(A_i : i \in E)$ *amalgamates in* $\mathcal{C}$.

Proof sketch.

Let $\mathcal{F}$ denote the set of $B = (B_i : i \in E)$ where $B_i \in \mathcal{C}$, $B_i \subset A_i$, and B_i is finitely generated. Partially $\mathcal{F}$ by $B \leq B'$ iff $B_i \subset B'_i$ for all i. Let $\mathcal{U}$ be an ultrafilter on $\mathcal{F}$ such that $\{X \mid X \geq B\} \in \mathcal{U}$ for all B. By hypothesis, there are embeddings $\phi_{B,i} \colon B_i \to D_B$ in $\mathcal{C}$ for each $B \in \mathcal{F}$. Let D be the ultraproduct $\prod_B D_B/\mathcal{U}$. Then $x \mapsto (\phi_{B,i}(x) : x \in B_i)/\mathcal{U}$ defines an embedding from A_i to D. □

Binary amalgamation can fail.

- Let $F_1 \cong F_2 \cong \mathbb{C}$ and $F_1 \cap F_2 = \mathbb{R}$. Then F_1 and F_2 do not amalgamate as integral domains.
- (Kimura, 1957) There are overlapping finite commutative semigroups S_1, S_2 that do not amalgamate as semigroups.
- It follows that there are two finite commutative rings that do not amalgamate as rings.
- (Sapir, 1997; Jackson, 2000) There is no algorithm that can decide whether two arbitrary finite semigroups amalgamate. Likewise for finite rings.
- There are many papers about binary amalgamation in categories of groups with various extra properties. Some have binary amalgamation. Some don't.

Integral domains

Let $F_1 = \mathbb{R}[x]/(x^2+1)$ and $F_2 = \mathbb{R}[y]/(y^2+1)$.

Any commutative ring R containing $F_1 \cup F_2$ also contains $S = \mathbb{R}[x,y]/(x^2+1, y^2+1)$.

But S has divisors of zero:

$x \pm y$ are not in the ideal generated by x^2+1, y^2+1.

But $(x+y)(x-y) = (x^2+1) - (y^2+1)$ is in that ideal.

Semigroups

Define overlapping commutative semigroups $S_i = \{0, a, b, c_i\}$ for $i = 1, 2$ as follows.

$\cdot$	c_2	0	a	b	c_1
c_2	c_2	0	b	b	
0	0	0	0	0	0
a	b	0	0	0	a
b	b	0	0	0	a
c_1		0	a	a	c_1

If some semigroup T contained $S_1 \cup S_2$, then we would reach the contradiction $a = c_1 b = c_1(ac_2) = (c_1 a)c_2 = ac_2 = b$.

Beyond sunflowers

- Overlapping vector spaces over a fixed field can always be amalgamated.
- (H. Neumann, 1948) There are three overlapping groups that do not amalgamate as groups.
- (H. Neumann, 1951) Any three overlapping abelian groups amalgamate as abelian groups.
- (H. Neumann, 1954) But there are four overlapping abelian groups that do not amalgamate as groups.
- Call a ring Boolean if every element x is idempotent: $x^2 = x$. It is known that sunflowers amalgamate in the category of Boolean rings. But general ternary amalgamation fails:
  - Let $xy = y$ generate Boolean ring A_1.
  - Let $yz = z$ generate Boolean ring A_2.
  - Let $zx = x$ generate Boolean ring A_3.
  - If A_1, A_2, A_3 amalgamated, even as commutative multiplicative semigroups, then $x = zx = (yz)x = y(zx) = yx = xy = y$.

Vector spaces

General vector space amalgamation succeeds because every subspace is a direct summand.

For example, consider overlapping vector spaces U_1, U_2, U_3. We construct W_3 containing $\bigcup_i U_i$ as follows.

1. $W_1 = U_1$
2. $U_2 = (U_2 \cap W_1) \oplus V_2$
3. $W_2 = W_1 \oplus V_2 \supset U_1 \cup U_2$
4. $U_3 = (U_3 \cap W_2) \oplus V_3$
5. $W_3 = W_2 \oplus V_3 \supset U_1 \cup U_2 \cup U_3$

Set-theoretic motivations I

First, some basics about cardinality:

A set S is countable or satisfies $|S| \leq \aleph_0$ if there is a strict linear order $<_S$ such that every proper initial segment $\{x \mid x <_S y\}$ is finite.

A set satisifes $|S| \leq \aleph_{n+1}$ if there is a strict linear order $<_S$ such that every proper initial segment $I = \{x \mid x <_S y\}$ satisfies $|I| \leq \aleph_n$.

Set-theoretic motivations II

A Boolean ring is projective if is a retract of a free Boolean ring.

A family $\mathcal{F}$ of countable subsets of a set S is a club if every countable $T \subset S$ is contained in some $E \in \mathcal{F}$ and $\mathcal{F}$ is closed with respect to unions of countable chains.

A subring A of a ring B is relatively complete if, for every principal ideal I of B, the ideal $I \cap A$ of A is also principal.

Theorem (M., 2016)

A Boolean ring B satisfying $|B| \leq \aleph_d$ is projective iff there it has the $(d+1)$-ary Freese-Nation property, that is, has a club $\mathcal{F}$ of subsets such that the any subring A of B generated by the union of at most d elements of $\mathcal{F}$ is relatively complete.

Set-theoretic motivations III

A family $\mathcal{F}$ of sets is directed if for all $X, Y \in \mathcal{F}$ there exists $Z \in \mathcal{F}$ such that $X \cup Y \subset Z$.

Lemma

If $\mathcal{F}$ is directed, every $X \in \mathcal{F}$ is countable, and $|\bigcup \mathcal{F}| \geq \aleph_3$, then there $X_1, X_2, X_3 \in \mathcal{F}$ that are not a sunflower.

I wanted to prove that the d-ary and $(d+1)$-ary Freese-Nation properties are inequivalent. I succeeded, but for $d \geq 3$, my proof involved cooking up a tricky Boolean algebra of cardinality $\aleph_d$ as a union of a directed family of countable Boolean algebras.

So, I had to find a safe harbor, avoiding all obstructions to higher-order Boolean amalgamation...

Higher-order pushouts

Henceforth assume that $\mathcal{C}$, a category consisting of a class of algebraic structures and all homomorphisms between them, has the following closure properties.

- If $A \in \mathcal{C}$ and $A \cong B$, then $B \in \mathcal{C}$.
- All finite diagrams in $\mathcal{C}$ have colimits.

Say $A_1, \ldots, A_n \in \mathcal{C}$ are $\mathcal{C}$-overlapping if $\bigcap_{i \in s} A_i \in \mathcal{C}$ for all $s \subset \{1, \ldots, n\}$.

Given $\mathcal{C}$-overlapping $A_1, \ldots A_n \in \mathcal{C}$, define the pushout of $A_1, \ldots, A_n$ to be the colimit of the diagram consisting of the morphisms $\mathrm{id}\colon \bigcap_{i \in t} A_i \to \bigcap_{i \in s} A_i$ for $s \subset t$.

Typically, there is canonical pushout of $A_1, \ldots A_n$. It is the algebra $B \in \mathcal{C}$ freely generated by the set of elements $\bigcup_i A_i$ and the set of relations $\bigcup_i \mathcal{R}_i$ where $\mathcal{R}_i$ is the set of all relations true of A_i.

A sufficient condition for amalgamation

Theorem (M.)

Suppose that:

- $\mathcal{C}$ *has binary amalgamation.*
- $A_1, \ldots, A_n$ *are* $\mathcal{C}$*-overlapping.*
- *The pushout of* $A_1 \cap A_i, \ldots, A_{i-1} \cap A_i$ *naturally embeds in* A_i *and in the pushout of* $A_1, \ldots A_{i-1}$*, for all* $i \leq n$*.*
- *In the pushout of* $A_1, \ldots, A_{i-1}$*, the intersection of* A_j *and the pushout of* $A_1 \cap A_i, \ldots, A_{i-1} \cap A_i$ *equals* $A_j \cap A_i$*, for all* $j < i \leq n$*.*

Then $A_1, \ldots, A_n$ *amalgamates in* $\mathcal{C}$*.*