Chains and the chain rule

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1 Chains

Say we want to find $\frac{d}{dx}\ln(x^2+5)$. We need to use the chain rule. To better see how to do this, let's write down a chain:

$$x \mapsto x^2+5 \mapsto \ln(x^2+5).$$

This chain of arrows represents the order of operations in computing $\ln(x^2+5)$. Given input x, we first square it and add 5, and second we take the natural logarithm of that. Define helper variables $w = x^2+5$ and $u = \ln w = \ln(x^2+5)$. Now our chain can be written $x \mapsto w \mapsto u$.

Why break things up into these two steps? Because these two steps correspond to functions that are easy to differentiate. $\frac{dw}{dx}$ and $\frac{du}{dw}$ are both easy: $\frac{dw}{dx} = 2x$ and $\frac{du}{dw} = 1/w$. Now put it all together: $u = \ln w = \ln(x^2+5)$ and

$$\frac{d}{dx}\ln(x^2+5) = \frac{du}{dx} = \frac{du}{dw}\frac{dw}{dx} = \frac{1}{w}(2x) = \frac{1}{x^2+5}(2x) = \frac{2x}{x^2+5}.$$

The chain rule is just $\frac{du}{dx} = \frac{du}{dw}\frac{dw}{dx}$. The short chain $x \mapsto u$ is too hard to differentiate directly, so we broke it up into $x \mapsto w \mapsto u$ and differentiated each link in the chain.

Here's another example. Let $y = ((x\ln x)^5+4)^{10}$. Let's try to find $\frac{dy}{dx}$. This time the chain is $x \mapsto x\ln x \mapsto (x\ln x)^5+4 \mapsto ((x\ln x)^5+4)^{10}$ because each stage is easy to differentiate. Let's use helper variables again: $v = x\ln x$ and $z = v^5+4 = (x\ln x)^5+4$. Then $y = z^{10}$ and our chain is $x \mapsto v \mapsto z \mapsto y$. For a chain of three arrows, the chain rule is:

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dv}\frac{dv}{dx}.$$

(Imagine the dz's and dv's cancelling each other to help you remember the pattern.) So, to find $\frac{dy}{dx}$, we just need to find $\frac{dy}{dz}$, $\frac{dz}{dv}$, and $\frac{dv}{dx}$. The first two are easy applications of the power rule: $\frac{dy}{dz} = 10z^9 = 10((x\ln x)^5+4)^9$ and $\frac{dz}{dv} = 5v^4 = 5(x\ln x)^4$. To find $\frac{dv}{dx}$, we use the product rule:

$$\frac{dv}{dx} = \frac{d}{dx}(x\ln x) = (\ln x)\frac{d}{dx}x + x\frac{d}{dx}\ln x = (\ln x)(1) + x(1/x) = (\ln x)+1.$$

Putting it all together, we have

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dv}\frac{dv}{dx} = \left[10((x\ln x)^5+4)^9\right]\left[5(x\ln x)^4\right]\left[(\ln x)+1\right].$$

2 Implicit Differentiation

Suppose we have the equation $\mathrm{e}^{y^3} = \ln(x+y)$ and we're asked to find $\frac{dx}{dy}$ as a function of x and y. The solution method is to differentiate both sides of the equation with to respect x and then solve for $\frac{dy}{dx}$:

$$\frac{d}{dx}\mathrm{e}^{y^3} = \frac{d}{dx}\ln(x+y)$$

Let's start with $\frac{d}{dx}\mathrm{e}^{y^3}$. The chain is $x \mapsto y \mapsto y^3 \mapsto \mathrm{e}^{y^3}$. (The idea behind the $x \mapsto y$ part is that if we zoom in on a small piece of the curve defined by the equation $\mathrm{e}^{y^3} = \ln(x+y)$, then that small piece passes the vertical line test, so y is locally a function of x.) Let $u = y^3$ and $v = \mathrm{e}^u = \mathrm{e}^{y^3}$, making our chain $x \mapsto y \mapsto u \mapsto v$. Then the chain rule yields

$$\frac{d}{dx}\mathrm{e}^{y^3} = \frac{dv}{dx} = \frac{dv}{du}\frac{du}{dy}\frac{dy}{dx} = \mathrm{e}^u(3y^2)\frac{dy}{dx} = \mathrm{e}^{y^3}(3y^2)\frac{dy}{dx}.$$

For the other side of the equation, the chain is $x \mapsto x+y \mapsto \ln(x+y)$. Let $w = x+y$ and $z = \ln w = \ln(x+y)$, making our chain $x \mapsto w \mapsto z$. Therefore, we have

$$\frac{d}{dx}\ln(x+y) = \frac{dz}{dx} = \frac{dz}{dw}\frac{dw}{dx} = \frac{1}{w}\left(1 + \frac{dy}{dx}\right) = \frac{1+\frac{dy}{dx}}{x+y}.$$

Since $\frac{d}{dx}\mathrm{e}^{y^3} = \frac{d}{dx}\ln(x+y)$, we have

$$\mathrm{e}^{y^3}(3y^2)\frac{dy}{dx} = \frac{1+\frac{dy}{dx}}{x+y}.$$

We just need to solve for $\frac{dy}{dx}$.

$$\begin{aligned}
(x+y)\mathrm{e}^{y^3}(3y^2)\frac{dy}{dx} &= 1 + \frac{dy}{dx} \\
(x+y)\mathrm{e}^{y^3}(3y^2)\frac{dy}{dx} &= 1 + \frac{dy}{dx} \\
(x+y)\mathrm{e}^{y^3}(3y^2)\frac{dy}{dx} - \frac{dy}{dx} &= 1 \\
\left((x+y)\mathrm{e}^{y^3}(3y^2) - 1\right)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{(x+y)\mathrm{e}^{y^3}(3y^2) - 1}
\end{aligned}$$