Highlights from linear algebra

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1 Systems of equations

A **leading entry** in a matrix is the first (leftmost) nonzero entry of a row. For example, the leading entries in the matrix below have values 5, 4, and -9.

$$\begin{bmatrix} 0 & 5 & 7 & 0 & 0 \\ 4 & 3 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9 & 0 \end{bmatrix}$$

A matrix is in **echelon form** if the leading entries move strictly to the right as the rows descend, and the all-zero rows are at the bottom. The above matrix is not in echelon form; the matrix below is.

$$\begin{bmatrix} 1 & 5 & 1 & 2 & 8 & 1 \\ 0 & 3 & 4 & 1 & 0 & 2 \\ 0 & 0 & 0 & 5 & 9 & 1 \\ 0 & 0 & 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If a matrix is in echelon form, then the columns containing leading entries are called **pivot columns**. In the above matrix, the first, second, fourth and fifth columns are pivot columns.

Suppose we are given the following system of equations.

$$\begin{aligned} c_1 + 5c_2 + c_3 + 2c_4 + 8c_5 + c_6 &= 1 \\ -c_1 - 2c_2 + 3c_3 - c_4 - 8c_5 + c_6 &= 1 \\ c_1 - 7c_2 - 15c_3 + 3c_4 + 17c_5 - 6c_6 &= -3 \\ -2c_1 - 4c_2 + 6c_3 - 7c_4 - 19c_5 + 8c_6 &= 6 \\ c_1 + 8c_2 + 5c_3 + 13c_4 + 20c_5 - 2c_6 &= 3 \\ -3c_1 - 15c_2 - 3c_3 - c_4 - 3c_5 + 12c_6 &= 17 \end{aligned}$$

The **augmented coefficient matrix** of this system is:

$$\begin{bmatrix} 1 & 5 & 1 & 2 & 8 & 1 & 1 \\ -1 & -2 & 3 & -1 & -8 & 1 & 1 \\ 1 & -7 & -15 & 3 & 17 & -6 & -3 \\ -2 & -4 & 6 & -7 & -19 & 8 & 6 \\ 1 & 8 & 5 & 13 & 20 & -2 & 3 \\ -3 & -15 & -3 & -1 & -3 & 12 & 17 \end{bmatrix}$$

After performing **elementary row operations**, we get the following echelon matrix.

$$\begin{bmatrix} 1 & 5 & 1 & 2 & 8 & 1 & 1 \\ 0 & 3 & 4 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 & 9 & 1 & 4 \\ 0 & 0 & 0 & 0 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that the bottom two rows correspond to the equation $0c_1 + 0c_2 + 0c_3 + 0c_4 + 0c_5 + 0c_6 = 0$, which is trivially true. On the other hand, a row of the form $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ would correspond to $0c_1 + 0c_2 + 0c_3 + 0c_4 + 0c_5 + 0c_6 = 5$, which is trivially false. If we ever get a row like that, then we know immediately that our original system of equations has no solutions.

Our above echelon matrix has four pivot columns: the first, second, fourth, and fifth columns. Therefore, c_3 and c_6 are **free variables**. (Note that the last column does not correspond to a free variable.) Letting s and t be arbitrary real numbers, we set $c_3 = s$ and $c_6 = t$ and then use back-substitution to find the general solution to our original system of equations. We start with the bottom nonzero row, solving $6c_5 + 7t = 8$ for c_5, and then proceed upwards, eventually computing the following.

$$\begin{aligned} c_1 &= -\frac{187}{15} + \frac{17}{3}s + \frac{347}{15}t \\ c_2 &= \frac{6}{5} - \frac{4}{3}s - \frac{19}{15}t \\ c_3 &= s \\ c_4 &= -\frac{8}{5} + \frac{19}{15}t \\ c_5 &= \frac{4}{3} - \frac{7}{6}t \\ c_6 &= t \end{aligned}$$

A system of equations with free variables always has either infinitely many solutions or no solutions. The no-solution case occurs exactly when we have a "bad row" like $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$.

Consider the following system of equations.

$$\begin{aligned} x + y + z &= 3 \\ y + z &= 8 \\ x + 2y + 2z &= 11 \\ -y + z &= 4 \end{aligned}$$

After performing elementary row operations on the augmented coefficient matrix to get it into echelon form, we have:

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three columns, which correspond to our variables x, y, z, are all pivot columns. Therefore, there are no free variables. (Note that the last column does not correspond to a free variable. Also, the row of all zeroes does not imply there is a free variable.) The bottom row corresponds to the equation $0x + 0y + 0z = 0$, which is trivially true. Therefore, using back-substitution, we find that our original system of equations has the unique solution $x = -5$, $y = 2$, $z = 6$.

A system of equations with no free variables always has either a unique solution or no solutions. The no-solution case occurs exactly when we have a "bad row" like $\begin{bmatrix} 0 & 0 & 0 & -7 \end{bmatrix}$.

A matrix is in **reduced row echelon form** if it is in echelon form and the leading entries all have value 1 and the leading entries have only zeroes directly above them. For example, the following matrix is in reduced echelon form.

$$\begin{bmatrix} 1 & 0 & 5 & 0 & 0 & 1 \\ 0 & 1 & 7 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Consider the echelon matrix we computed earlier:

$$\begin{bmatrix} 1 & 5 & 1 & 2 & 8 & 1 & 1 \\ 0 & 3 & 4 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 & 9 & 1 & 4 \\ 0 & 0 & 0 & 0 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

After performing more elementary row operations, we can get it into reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & -\frac{17}{3} & 0 & 0 & -\frac{347}{15} & -\frac{187}{15} \\ 0 & 1 & \frac{4}{3} & 0 & 0 & \frac{19}{15} & \frac{6}{5} \\ 0 & 0 & 0 & 1 & 0 & -\frac{19}{15} & -\frac{8}{5} \\ 0 & 0 & 0 & 0 & 1 & \frac{7}{6} & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that putting an augmented coefficient matrix in reduced row echelon form makes back-substition much, much easier.

2 Linear dependence

If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are vectors, then they are **linearly dependent** if and only if there exists real numbers $c_1, \ldots, c_k$ not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$. How do we compute whether linear dependence occurs or not? Let's proceed by example. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_5$ are the following vectors in $\mathbb{R}^6$.

$$\begin{aligned}
\mathbf{v}_1 &= (1, -1, 1, -2, 1, -3) \\
\mathbf{v}_2 &= (5, -2, -7, -4, 8, -15) \\
\mathbf{v}_3 &= (1, 3, -15, 6, 5, -3) \\
\mathbf{v}_4 &= (1, -1, 3, -7, 13, -1) \\
\mathbf{v}_5 &= (8, -8, 17, -19, 20, -3)
\end{aligned}$$

Let's write these vectors as columns.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -2 \\ -7 \\ -4 \\ 8 \\ -15 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -15 \\ 6 \\ 5 \\ -3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -7 \\ 13 \\ -1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 8 \\ -8 \\ 17 \\ -19 \\ 20 \\ -3 \end{bmatrix}$$

So, are $\mathbf{v}_1, \ldots, \mathbf{v}_5$ linearly dependent? Let's write out the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ in terms of column vectors.

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \\ 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -2 \\ -7 \\ -4 \\ 8 \\ -15 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 3 \\ -15 \\ 6 \\ 5 \\ -3 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ -1 \\ 3 \\ -7 \\ 13 \\ -1 \end{bmatrix} + c_5 \begin{bmatrix} 8 \\ -8 \\ 17 \\ -19 \\ 20 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly, $c_1 = c_2 = c_3 = c_4 = c_5 = 0$ is a solution, but is it the only solution? If there's another solution, then $\mathbf{v}_1, \ldots, \mathbf{v}_5$ are linearly dependent. If $c_1 = c_2 = c_3 = c_4 = c_5 = 0$ is the unique solution, then $\mathbf{v}_1, \ldots, \mathbf{v}_5$ are **linearly independent**. To determine whether there is another solution, we just need to check if any of $c_1, \ldots, c_5$ is a free variable. The corresponding augmented coefficient matrix is:

$$\begin{bmatrix} 1 & 5 & 1 & 2 & 8 & 0 \\ -1 & -2 & 3 & -1 & -8 & 0 \\ 1 & -7 & -15 & 3 & 17 & 0 \\ -2 & -4 & 6 & -7 & -19 & 0 \\ 1 & 8 & 5 & 13 & 20 & 0 \\ -3 & -15 & -3 & -1 & -3 & 0 \end{bmatrix}$$

After performing elementary row operations, we get the following echelon matrix.

$$\begin{bmatrix} 1 & 5 & 1 & 2 & 8 & 0 \\ 0 & 3 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 9 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We immediately see that c_3 is a free variable. Therefore, $\mathbf{v}_1, \dots, \mathbf{v}_5$ are linearly dependent. For example, if we set c_3 equal to any nonzero value, say, 3, then, using back substitution, we find that $c_5 = 0$, $c_4 = 0$, $c_2 = -4$, and $c_1 = 17$, so $(c_1, c_2, c_3, c_4, c_5) = (17, -4, 3, 0, 0)$ is a solution. So, the equation $17\mathbf{v}_1 - 4\mathbf{v}_2 + 3\mathbf{v}_3 = \mathbf{0}$ witnesses that $\mathbf{v}_1, \dots, \mathbf{v}_5$ are linearly dependent (and that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent).

Remember: linear dependence is equivalent to having free variables.

3 Span

The **span** of a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is the set of all **linear combinations** $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ of those vectors. For example, $3\mathbf{v}_1 - 4\mathbf{v}2 + \mathbf{v}_4$, $\mathbf{0}$, and $-11\mathbf{v}_5$ are each in the span of $\{\mathbf{v}_1, \dots, \mathbf{v}_5\}$.

Suppose we want to determine whether $\mathbf{w}$ is in the span of $\mathbf{v}_1, \dots, \mathbf{v}_6$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -2 \\ -7 \\ -4 \\ 8 \\ -15 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -15 \\ 6 \\ 5 \\ -3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -7 \\ 13 \\ -1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 8 \\ -8 \\ 17 \\ -19 \\ 20 \\ -3 \end{bmatrix}, \quad \mathbf{v}_6 = \begin{bmatrix} 1 \\ 1 \\ -6 \\ 8 \\ -2 \\ 12 \end{bmatrix},$$

$$\text{and } \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 6 \\ 3 \\ 17 \end{bmatrix}.$$

Then we just need to check whether $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{w}$ has any solutions. The augmented coefficent matrix, which you may recognize from earlier, is:

$$\begin{bmatrix} 1 & 5 & 1 & 2 & 8 & 1 & 1 \\ -1 & -2 & 3 & -1 & -8 & 1 & 1 \\ 1 & -7 & -15 & 3 & 17 & -6 & -3 \\ -2 & -4 & 6 & -7 & -19 & 8 & 6 \\ 1 & 8 & 5 & 13 & 20 & -2 & 3 \\ -3 & -15 & -3 & -1 & -3 & 12 & 17 \end{bmatrix}$$

After performing elementary row operations, we again get the following echelon matrix.

$$\begin{bmatrix} 1 & 5 & 1 & 2 & 8 & 1 & 1 \\ 0 & 3 & 4 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 & 9 & 1 & 4 \\ 0 & 0 & 0 & 0 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are no "bad rows" like $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 13 \end{bmatrix}$, we conclude that there is a solution. Therefore, $\mathbf{w}$ is in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_6$. For example, if we set our free variables c_3 and c_6 both equal to 0, then $c_1 = -187/15$, $c_2 = 6/5$, $c_4 = -8/5$, and $c_5 = 4/3$. Therefore, $-\frac{187}{15}\mathbf{v}_1 + \frac{6}{5}\mathbf{v}_2 - \frac{8}{5}\mathbf{v}_4 + \frac{4}{3}\mathbf{v}_5 = \mathbf{w}$.

4 Bases

Roughly speaking, a **basis** of a vector space is a (linear) coordinate system. For example, in $\mathbb{R}^3$, every vector is uniquely determined by its first, second, and third coordinates. Therefore, every vector (p, q, r) in $\mathbb{R}^3$ is equal to a unique linear combination of the vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = p \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + q \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$$

In general, a set of vectors $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis of a vector space V if, for every vector $\mathbf{w}$, the equation $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{w}$ has a unique solution.

If A and B are both bases of V, then A and B have the same size. We call this size the **dimension** of V (written $\dim(V)$). For example, $\dim(\mathbb{R}^3) = 3$.

How can we tell whether or not a set B is a basis of V? It turns out that B is a basis of V if and only if any of the following equivalent conditions hold.

1. B has size $\dim(V)$ and B is linearly independent. (If we know what $\dim(V)$ is, then this condition is usually easier to check than the others.)
2. B has size $\dim(V)$ and the span of B is V.
3. B is linearly independent and the span of B is V.
4. B is a maximal linearly independent subset of V (*i.e.*, B is linearly independent but if we add any other element of V to B, then B will become linearly dependent).
5. B is a minimal spanning set for V (*i.e.*, the span of B is V, but if we remove any element of B, then the span of B will become smaller than V).

For example, a set B of vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$ if and only if B has size n and B is linearly independent. If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then as before, we can check whether B is linearly independent by writing out the augmented coefficient matrix for the equation $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$, putting it into echelon form, and checking for free variables.

5 Determinants

For every square matrix A we can define a number $\det(A)$ called the **determinant** of A. It has nice properties:

1. The determinant of A is nonzero if and only if the columns of A are linearly independent.
2. The determinant of A is nonzero if and only if the rows of A are linearly independent.
3. The determinant of A is nonzero if and only if when A is put into echelon form, all columns are pivot columns.
4. The determinant of A is nonzero if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only one solution: $\mathbf{x} = \mathbf{0}$.
5. The determinant of A is nonzero if and only if for every $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for $\mathbf{x}$.

6 Subspaces

Abstractly, a set W of vectors in a vector space V is a **subspace** of V if for all $\mathbf{x}, \mathbf{y}$ in W and all c in $\mathbb{R}$, the vectors $\mathbf{x}+\mathbf{y}$ and $c\mathbf{x}$ are also in W. Geometrically, the subspaces of $\mathbb{R}^2$ are $\{\mathbf{0}\}$, lines that pass through the origin, and $\mathbb{R}^2$; the subspaces of $\mathbb{R}^3$ are $\{\mathbf{0}\}$, lines that pass through the origin, planes that pass through the origin, and $\mathbb{R}^3$.

Subspaces are vector spaces too, so they have bases and dimensions. For example, given a plane P that passes through the origin, any two non-collinear vectors in P form a basis of P. Therefore, P has dimension 2.

Given any set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors in a vector space V, the span of S is a subspace of V. For example, given any two non-collinear vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the span of $\{\mathbf{u}, \mathbf{v}\}$ is a plane that passes through the origin.

If A is an $m \times n$ matrix, then the span of the rows of A is a subspace of $\mathbb{R}^n$. We call this the **row space** of A, and denote it by $\mathrm{Row}(A)$. Similarly, the span of the columns of A, which we call the **column space** of A and denote by $\mathrm{Col}(A)$, is a subspace of $\mathbb{R}^m$. For example, consider the 6×5 matrix A below.

$$A = \begin{bmatrix} 1 & 5 & 1 & 2 & 8 \\ -1 & -2 & 3 & -1 & -8 \\ 1 & -7 & -15 & 3 & 17 \\ -2 & -4 & 6 & -7 & -19 \\ 1 & 8 & 5 & 13 & 20 \\ -3 & -15 & -3 & -1 & -3 \end{bmatrix}$$

The row space of A is the subspace of $\mathbb{R}^5$ spanned by the six vectors $(1, 5, 1, 2, 8)$, $(-1, -2, 3, -1, -8)$, $(1, -7, -15, 3, 17)$, $(-2, -4, 6, -7, -19)$, $(1, 8, 5, 13, 20)$, and $(-3, -15, -3, -1, -3)$. The column space of A is the subspace of $\mathbb{R}^6$ spanned by the five vectors $(1,-1,1,-2,1,-3)$, $(5,-2,-7,-4,8,-15)$, $(1,3,-15,6,5,-3)$, $(1,-1,3,-7,13,-1)$, and $(8,-8,17,-19,20,-3)$.

There is an algorithm for computing a basis of a row space. For example, consider the above matrix A. Use elementary row operations to turn A into an echelon matrix E:

$$E = \begin{bmatrix} 1 & 5 & 1 & 2 & 8 \\ 0 & 3 & 4 & 1 & 0 \\ 0 & 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows of E, which are $(1,5,1,2,8)$, $(0,3,4,1,0)$, $(0,0,0,5,9)$, and $(0,0,0,0,6)$, form a basis or $\text{Row}(A)$. Therefore, $\text{Row}(A)$ is a 4-dimensional subspace of $\mathbb{R}^5$.

We can also use E to compute a basis of $\text{Col}(A)$. The columns of A corresponding to pivot columns of E form a basis of $\text{Col}(A)$. In other words, since the first, second, fourth, and fifth columns of E are pivot columns, the first, second, fourth, and fifth columns of A form a basis of $\text{Col}(A)$. Specifically, $(1,-1,1,-2,1,-3)$, $(5,-2,-7,-4,8,-15)$, $(2,-1,3,-7,13,-1)$, and $(8,-8,17,-19,20,-3)$ form a basis of $\text{Col}(A)$. Therefore, $\text{Col}(A)$ is a 4-dimensional subspace of $\mathbb{R}^6$.

It is no accident that $\text{Row}(A)$ and $\text{Col}(A)$ have the same dimension. The dimension of $\text{Row}(A)$ equals the number of nonzero rows of E. The dimension of $\text{Col}(A)$ equals the number of pivot columns of E. But in any echelon matrix, every pivot column corresponds to a unique leading entry, and every leading entry corresponds to a unique nonzero row, so there are exactly as many pivot columns as nonzero rows. We call the common dimension of $\text{Row}(A)$ and $\text{Col}(A)$ the **rank** of A.

Given any $m \times n$ matrix B, the set of all $\mathbf{x}$ in $\mathbb{R}^n$ satisfying $B\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$. We call this subspace the **null space** of B and denote it by $N(B)$.

Consider the following 6×6 matrix B.

$$\begin{bmatrix} 1 & 5 & 1 & 2 & 8 & 1 \\ -1 & -2 & 3 & -1 & -8 & 1 \\ 1 & -7 & -15 & 3 & 17 & -6 \\ -2 & -4 & 6 & -7 & -19 & 8 \\ 1 & 8 & 5 & 13 & 20 & -2 \\ -3 & -15 & -3 & -1 & -3 & 12 \end{bmatrix}$$

We can compute its null space by adjoining a column of zeroes to it in order to get the following augmented coefficient matrix for $B\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & 5 & 1 & 2 & 8 & 1 & 0 \\ -1 & -2 & 3 & -1 & -8 & 1 & 0 \\ 1 & -7 & -15 & 3 & 17 & -6 & 0 \\ -2 & -4 & 6 & -7 & -19 & 8 & 0 \\ 1 & 8 & 5 & 13 & 20 & -2 & 0 \\ -3 & -15 & -3 & -1 & -3 & 12 & 0 \end{bmatrix}$$

After performing elementary row operations, we get the following echelon matrix.

$$\begin{bmatrix} 1 & 5 & 1 & 2 & 8 & 1 & 0 \\ 0 & 3 & 4 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 & 9 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is:

$$\begin{aligned} x_1 + 5x_2 + x_3 + 2x_4 + 8x_5 + x_6 &= 0 \\ 3x_2 + 4x_3 + x_4 + 2x_6 &= 0 \\ 5x_4 + 9x_5 + x_6 &= 0 \\ 6x_5 + 7x_6 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

The free variables are x_3 and x_6. If s and t are arbitrary reals, then the general solution to $B\mathbf{x} = \mathbf{0}$, as computed by back-substitution, is:

$$\begin{aligned} x_1 &= \frac{17}{3}s + \frac{347}{15}t \\ x_2 &= -\frac{4}{3}s - \frac{19}{15}t \\ x_3 &= s \\ x_4 &= \frac{19}{15}t \\ x_5 &= -\frac{7}{6}t \\ x_6 &= t \end{aligned}$$

We can rewrite this solution as follows.

$$\mathbf{x} = \begin{bmatrix} \frac{17}{3}s + \frac{347}{15}t \\ -\frac{4}{3}s - \frac{19}{15}t \\ s \\ \frac{19}{15}t \\ -\frac{7}{6}t \\ t \end{bmatrix} = s \begin{bmatrix} \frac{17}{3} \\ -\frac{4}{3} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{347}{15} \\ -\frac{19}{15} \\ 0 \\ \frac{19}{15} \\ -\frac{7}{6} \\ 1 \end{bmatrix}$$

The last two vectors, $(\frac{17}{3}, -\frac{4}{3}, 1, 0, 0, 0)$ and $(\frac{347}{15}, -\frac{19}{15}, 0, \frac{19}{15}, -\frac{7}{6}, 1)$, form a basis of $N(B)$. Therefore, the null space of B is a 2-dimensional subspace of $\mathbb{R}^6$.

A similar computation shows that the general solution for $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} \frac{17}{3}s \\ -\frac{4}{3}s \\ s \\ 0 \\ 0 \end{bmatrix} = s \begin{bmatrix} \frac{17}{3} \\ -\frac{4}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

where s is an arbitrary real. (Notice that now $\mathbf{x}$ is in $\mathbb{R}^5$ because A is a 6×5 matrix.) The single vector $(\frac{17}{3}, -\frac{4}{3}, 1, 0, 0)$ forms a basis of $N(A)$; hence, $N(A)$ is a 1-dimensional subspace of $\mathbb{R}^5$.

Notice that $\dim(N(A)) + \text{rank}(A) = 1 + 4 = 5$, which is the number of columns in A. This is not an accident. In general, the dimension of $N(C)$ for any $m \times n$ matrix C equals the number of free variables in the equation $C\mathbf{x} = \mathbf{0}$, which equals the number of non-pivot columns in an echelon matrix obtained from C by elementary row operations. Since this echelon matrix has n columns, it must have $n - \dim(N(C))$ pivot columns. Since the rank of C equals the number of pivot columns in the echelon matrix, we have $\text{rank}(C) = n - \dim(N(C))$. Equivalently, we have $\text{rank}(C) + \dim(N(C)) = n$.