Highlights from linear algebra

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1 Example of the eigenvector method

Consider the following initial value problem.

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 4 & 0 \\ 0 & 5 & 2 \end{bmatrix} \vec{x}(t) \text{ and } \vec{x}(0) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

To solve this problem, we first find the general solution to the differential equation. Finding the general solution boils down to finding three linearly independent solutions $\vec{x}_1(t)$, $\vec{x}_2(t)$, and $\vec{x}_3(t)$, for the general solution is then just the an arbitrary linear combination $c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$ of these three solutions. Once we have our general solution $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$, we will use the initial condition $\vec{x}(0) = (3, 2, 1)$ to solve for c_1, c_2, and c_3.

To find three linearly independent solutions, we will use the eigenvector method. First, we must compute the eigenvalues of A. To do that we need to find the roots of the following determinant, which is a cubic function of λ.

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 5 & 2-\lambda \end{vmatrix}$$

Notice that the second row has only one nonzero entry. (Likewise for the first and third columns.) Therefore, factoring the cubic will be relatively easy if we expand about the second row:

$$-0\begin{vmatrix} -1 & 0 \\ 5 & 2 \end{vmatrix} + (4-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} - 0\begin{vmatrix} 2-\lambda & -1 \\ 0 & 4-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda)^2.$$

We have a non-repeated real eigenvalue 4 and a repeated real eigenvalue 2 with multiplicity 2. Therefore, there are three linearly independent solutions, one involving e^{4t}, and two involving e^{2t}.

First consider the eigenvalue 4. To get a solution to our differential equation corresponding to this eigenvalue, we need to compute the corresponding eigenbasis, which is just a basis of $N(A-4I)$, the null space of $A-4I$. Because 4 is a non-repeated eigenvalue, this null space has dimensiion 1. Therefore, finding the eigenbasis is the same as finding a single eigenvector, which is just a nontrivial solution $\vec{u}$ to $(A-4I)\vec{u} = \vec{0}$:

$$\begin{bmatrix} -2 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After performing a single elementary row operation, we get the following echelon system of equations.

$$\begin{bmatrix} -2 & -1 & 0 \\ 0 & 5 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Notice that u_3 is a free variable while u_1 and u_2 correspond to pivot columns. Since we want $\vec{u}$ to be nontrivial, that is, not $\vec{0}$, let us choose u_3 to be nonzero, say, $u_3 = 5$. Then we see that $u_2 = 2$ and $u_1 = -1$ by back-substitution. Therefore, $(-1, 2, 5)$ is an eigenvector of A with corresponding eigenvalue 4, and $\{(-1, 2, 5)\}$ is an eigenbasis corresponding eigenvalue 4. Therefore, we may choose $\vec{x}_1(t) = (-1, 2, 5)\mathrm{e}^{4t}$ to be one of our three linearly independent solutions to our differential equation.

Next consider the eigenvalue 2. We need to compute a corresponding eigenbasis, which is just a basis of $N(A - 2I)$. This null space has dimension at least 1, but it could be also be as high as 2, the multiplicity of the eigenvalue 2. (The case where the dimension is equal to the multiplicity is called the complete case, and is actually siginificantly easier to solve than the so-called defective case, in which the dimension is less than the multiplicity.) To find our eigenbasis, we must find the general solution to $(A - 2I)\vec{v} = \vec{0}$:

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After performing two elementary row operations, we get the following echelon system of equations.

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Notice that v_1 and v_3 are free variables while v_2 corresponds to a pivot column. Let $v_1 = r$ and $v_3 = s$. It is easily seen that $v_2 = 0$. Hence, the general solution for $\vec{v}$ is $(r, 0, s)$, which we rewrite as $r(1, 0, 0) + s(0, 0, 1)$ to deduce that $\{(1, 0, 0), (0, 0, 1)\}$ is a basis of $N(A - 2I)$. Therefore, $\{(1, 0, 0), (0, 0, 1)\}$ is an eigenbasis corresponding to the eigenvalue 2, so we may choose $\vec{x}_2(t) = (1, 0, 0)\mathrm{e}^{2t}$ and $\vec{x}_3(t) = (0, 0, 1)\mathrm{e}^{2t}$ to be the other two of our three linearly independent solutions to our differential equation.

Thus, the general solution to our differential equation is

$$\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \mathrm{e}^{4t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{2t}.$$

Finally, we must use $\vec{x}(0) = (3, 2, 1)$ to solve for c_1, c_2, and c_3:

$$\vec{x}(0) = c_1 \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} 1 + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} 1 + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} 1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Let us rewrite this equation in terms of matrix multiplication.

$$\begin{bmatrix} -1 & 1 & 0 \\ 2 & 0 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

After performing several elementary row operations, we have the following echelon system of equations.

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ -8 \end{bmatrix}$$

Using back-substitution, we find that $c_3 = -4$, $c_2 = 4$, and $c_1 = 1$. Therefore, the solution to our initial value problem is

$$\vec{x}(t) = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} e^{4t} + 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} - 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} -e^{4t} + 4e^{2t} \\ 2e^{4t} \\ 5e^{4t} - 4e^{2t} \end{bmatrix}.$$

Exercise 1. Solve the following initial value problem.

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}(t) \text{ and } \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

2 Complex eigenvalues

Consider the following differential equation.

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) = \begin{bmatrix} -6 & 0 & 5 \\ 7 & 3 & -8 \\ -4 & 0 & 2 \end{bmatrix} \vec{x}(t)$$

Let us find the general real solution. I include "real" because for this problem the eigenvector method naively applied only yields the general complex solution, which is too general for many real-world applications in which we only want real solutions.

As before, we find three real linearly independent solutions; the general real solution will be an arbitrary real linear combination of these three. To find these three solutions, we first find the eigenvalues of A by finding the roots of the following determinant.

$$\begin{vmatrix} -6-\lambda & 0 & 5 \\ 7 & 3-\lambda & -8 \\ -4 & 0 & 2-\lambda \end{vmatrix}$$

Expanding about the second column is by far the easiest way to compute this determinant:

$$-0 \begin{vmatrix} 7 & -8 \\ -4 & 2 \end{vmatrix} + (3-\lambda) \begin{vmatrix} -6-\lambda & 5 \\ -4 & 2-\lambda \end{vmatrix} - 0 \begin{vmatrix} -6-\lambda & 5 \\ 7 & -8 \end{vmatrix} = (3-\lambda)(\lambda^2 + 4\lambda + 8).$$

Clearly, 3 is one of the eigenvalues of A. By the quadratic formula or by completion of the square, we find that the other two eigenvalues are $-2 \pm 2i$.

First consider the eigenvalue 3. Solve $(A - 3I)\vec{u} = \vec{0}$:

$$\begin{bmatrix} -9 & 0 & 5 \\ 7 & 0 & -8 \\ -4 & 0 & -1 \end{bmatrix} \vec{u} = \vec{0}.$$

After performing three elementary row operations, we have the following echelon system of equations.

$$\begin{bmatrix} -9 & 0 & 5 \\ 0 & 0 & -37 \\ 0 & 0 & 0 \end{bmatrix} \vec{u} = \vec{0}$$

We see that u_2 is a free variable and that u_1 and u_3 correspond to pivot columns. By back substitution, we deduce that the general (real) solution for $\vec{u}$ is $(0, r, 0)$ where r is an arbitrary real. Therefore, $\{(0, 1, 0)\}$ is an eigenbasis corresponding to the eigenvalue 3, so we may choose one of our three linearly independent solutions to our differential equation to be $\vec{x}_1 = (0, 1, 0)e^{3t}$.

Next consider the eigenvalue $-2 + 2i$. Solve $(A - (-2 + 2i)I)\vec{v} = \vec{0}$:

$$\begin{bmatrix} -4-2i & 0 & 5 \\ 7 & 5-2i & -8 \\ -4 & 0 & 4-2i \end{bmatrix} \vec{v} = \vec{0}.$$

We proceed in the usual way, except using complex arithmetic. Perform the elementary row operations $R_2 \to R_2(4 + 2i) + 7R_1$ and $R_3 \to R_3(4 + 2i) - 4R_1$:

$$\begin{bmatrix} -4-2i & 0 & 5 \\ 0 & 24+2i & 3-16i \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \vec{0}.$$

Now our matrix is in echelon form; v_3 is a free variable while v_1 and v_2 correspond to pivot columns. The complex arithmetic makes performing the back-substitution a little harder, but we can still do it if we remember how to divide complex numbers:

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(a + bi)(c - di)}{c^2 + d^2}.$$

Eventually, we find that the general (complex) solution for $\vec{v}$ is

$$\vec{v} = \begin{bmatrix} -r - \frac{ri}{2} \\ -\frac{2r}{29} + i\frac{39r}{58} \\ r \end{bmatrix} = r \begin{bmatrix} -1 - \frac{i}{2} \\ -\frac{2}{29} + i\frac{39}{58} \\ 1 \end{bmatrix}$$

where r is an arbitrary complex number. Choose $r = 58$ to get rid of the fractions and get $\{(-58 - 29i, -4 + 39i, 58)\}$ as an eigenbasis corresponding to the eigenvalue $-2 + 2i$. Therefore, if we admitted complex solutions, then we could choose $(-58 - 29i, -4 + 39i, 58)e^{(-2+2i)t}$ to be the second of our three linearly independent solutions to our differential equation.

To get two linearly independent real solutions corresponding to our conjugate pair of complex eigenvalues $-2 \pm 2i$, it actually suffices to just take the real and imaginary parts of our complex solution $(-58-29i, -4+39i, 58)e^{(-2+2i)t}$:

$$\begin{bmatrix} -58-29i \\ -4+39i \\ 58 \end{bmatrix} e^{(-2+2i)t} = \begin{bmatrix} -58-29i \\ -4+39i \\ 58 \end{bmatrix} e^{-2t}(\cos(2t) + i\sin(2t))$$
$$= e^{-2t} \begin{bmatrix} -58\cos(2t) - 29i\cos(2t) - 58i\sin(2t) + 29\sin(2t) \\ -4\cos(2t) - 4i\sin(2t) + 39i\cos(2t) - 39\sin(2t) \\ 58\cos(2t) + 58i\sin(2t) \end{bmatrix}$$
$$= e^{-2t} \begin{bmatrix} -58\cos(2t) + 29\sin(2t) \\ -4\cos(2t) - 39\sin(2t) \\ 58\cos(2t) \end{bmatrix} + ie^{-2t} \begin{bmatrix} -29\cos(2t) - 58\sin(2t) \\ -4\sin(2t) + 39\cos(2t) \\ 58\sin(2t) \end{bmatrix}.$$

Therefore, we may choose the second and third of our three linearly independent solutions of our differential equation to be

$$\vec{x}_2(t) = e^{-2t} \begin{bmatrix} -58\cos(2t) + 29\sin(2t) \\ -4\cos(2t) - 39\sin(2t) \\ 58\cos(2t) \end{bmatrix} \text{ and } \vec{x}_3(t) = e^{-2t} \begin{bmatrix} -29\cos(2t) - 58\sin(2t) \\ -4\sin(2t) + 39\cos(2t) \\ 58\sin(2t) \end{bmatrix}.$$

Thus, the general real solution of our differential equation is $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$ where c_1, c_2, and c_3 are arbitrary reals:

$$\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -58\cos(2t) + 29\sin(2t) \\ -4\cos(2t) - 39\sin(2t) \\ 58\cos(2t) \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} -29\cos(2t) - 58\sin(2t) \\ -4\sin(2t) + 39\cos(2t) \\ 58\sin(2t) \end{bmatrix}.$$

Exercise 2. Find the general real solution to the following differential equation.

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -4 \\ -1 & 0 & 0 \end{bmatrix} \vec{x}$$

Hint: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.