NON-ABSOLUTENESS OF MODEL EXISTENCE AT $\aleph_\omega$

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Abstract. In [FHK13], the authors considered the question whether model-existence of $\mathcal{L}_{\omega_1,\omega}$-sentences is absolute for transitive models of ZFC, in the sense that if $V \subseteq W$ are transitive models of ZFC with the same ordinals, $\varphi \in V$ and $V \models$ "φ is an $\mathcal{L}_{\omega_1,\omega}$-sentence", then $V \models \Phi$ if and only if $W \models \Phi$ where Φ is a first-order sentence with parameters φ and α asserting that φ has a model of size $\aleph_\alpha$.

From [FHK13] we know that the answer is positive for $\alpha = 0, 1$ and under the negation of CH, the answer is negative for all $\alpha > 1$. Under GCH, and assuming the consistency of a supercompact cardinal, the answer remains negative for each $\alpha > 1$, except the case when $\alpha = \omega$ which is an open question in [FHK13].

We answer the open question by providing a negative answer under GCH even for $\alpha = \omega$. Our examples are incomplete sentences. In fact, the same sentences can be used to prove a negative answer under GCH for all $\alpha > 1$ assuming the consistency of a Mahlo cardinal. Thus, the large cardinal assumption is relaxed from a supercompact in [FHK13] to a Mahlo cardinal.

Finally, we consider the absoluteness question for the $\aleph_\alpha$-amalgamation property of $\mathcal{L}_{\omega_1,\omega}$-sentences (under *substructure*). We prove that assuming GCH, $\aleph_\alpha$-amalgamation is non-absolute for $1 < \alpha < \omega$. This answers a question from [SS]. The cases $\alpha = 1$ and α infinite remain open. As a corollary we get that it is non-absolute that the amalgamation spectrum of an $\mathcal{L}_{\omega_1,\omega}$-sentence is empty.

1. Introduction

The current paper adds to the literature that investigates which notions for infinitary logics, or more generally for abstract elementary classes, are absolute for models of ZFC. Some notions like satisfiability[a], model-existence in $\aleph_0$ and $\aleph_1$, model-existence in some $\kappa \geq \beth_{\omega_1}$, $\aleph_0$-amalgamation and $\aleph_0$-joint embedding are absolute between transitive models of ZFC (see [Bal12, FHK13, GS86]). Other notions such as model-existence in $\aleph_\alpha$, $\alpha > 1$, or existence of a maximum model in $\aleph_\alpha$, $\alpha > 1$ are non-absolute (see [FHK13, BKS16, BS]). Unfortunately, the absoluteness question remains open for a wide range of notions, such as $\aleph_1$-categoricity for $\mathcal{L}_{\omega_1,\omega}$-sentences, $\aleph_1$- amalgamation, and $\aleph_1$-joint embedding, to name a few.

The notions we consider in this paper are "model-existence" and "amalgamation". For $\aleph_0$ and $\aleph_1$, model existence is an absolute notion for transitive models of ZFC. From [Mal68], we know that there is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ that characterizes $2^{\aleph_0}$. That is, ϕ has models in all (infinite) cardinalities less or equal to $2^{\aleph_0}$, but no larger models, and this is a theorem of ZFC. Under CH, ϕ has models in $\aleph_1$, but no models in $\aleph_2$. Under the negation of CH, ϕ has a model in $\aleph_2$. Hence, model-existence in $\aleph_2$ is not an absolute notion.

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[a]We mean that given a model M and a sentence ϕ, the statement "$M \models \phi$" is absolute.

Similarly, other consistent violations of GCH witness that for each $1 < \alpha < \omega_1$ model existence in $\aleph_\alpha$ is not absolute.

Recall that $\beth_{\omega_1}$ is the Hanf number for $\mathcal{L}_{\omega_1,\omega}$. *I.e.*, every $\mathcal{L}_{\omega_1,\omega}$-sentence which has models in all cardinalities below $\beth_{\omega_1}$, it also has arbitrarily large models. By [GS86], the property that an $\mathcal{L}_{\omega_1,\omega}$-sentence has arbitrarily large models is absolute.

So, it is natural to ask whether model-existence in $\aleph_\alpha$, with $\aleph_1 < \aleph_\alpha < \beth_{\omega_1}$, is absolute for models of ZFC+GCH. The question was answered in [FHK13], under large cardinal assumptions, except the case where $\alpha = \omega$. The large cardinal assumptions are different for successors of successors than for limit cardinals and successors of limits.

The following result from [FHK13], shows that, assuming the consistency of uncountably many inaccessibles, model-existence in $\aleph_{\alpha+2}$ is a non-absolute notion for $\mathcal{L}_{\omega_1,\omega}$-sentences, for all $\alpha < \omega_1$.

Theorem 1.1 ([FHK13], Theorem 7)**.** *Assume a ground model V of ZFC+GCH in which there are uncountably many inaccessible cardinals and an inner model $M \subset V$ of ZFC+GCH and "$\diamondsuit^+_\kappa$ holds for every regular uncountable $\kappa < \aleph_{\omega_1}$." Then there is a generic extension $V[G]$ in which the GCH is true and model-existence in $\aleph_{\alpha+2}$ for $\mathcal{L}_{\omega_1,\omega}$-sentences is not absolute between M and $V[G]$ for all $\alpha < \omega_1^M$.*[b][c]

The way the proof of Theorem 1.1 goes is that for each $\alpha < \omega_1^M$ there exists an $\mathcal{L}_{\omega_1,\omega}$-sentence $\sigma^{\alpha+2}$ such that $\sigma^{\alpha+2}$ has a model of size $\aleph_{\alpha+2}$ if and only if there exists an $\aleph_{\alpha+1}$-Kurepa family. Moreover, this equivalence is absolute between transitive models of ZFC that contain $\sigma^{\alpha+2}$. It follows that $\sigma^{\alpha+2}$ has a model of size $\aleph_{\alpha+2}$ in M. In addition, since Levy collapsing an inaccessible to $\aleph_{\alpha+2}$ destroys all $\aleph_{\alpha+1}$-Kurepa trees, there is a generic extension of V where $\sigma^{\alpha+2}$ does not have any models of size $\aleph_{\alpha+2}$. So, another form of Theorem 1.1 is implicit in [FHK13]. Assuming M and V as in Theorem 1.1 except now with just one inaccessible cardinal, then, for each $\alpha < \omega_1^M$, model-existence in $\aleph_{\alpha+2}$ for $\mathcal{L}_{\omega_1,\omega}$-sentences is not absolute between M and a forcing extension $V[G_\alpha]$ satisfying GCH. This covers the case of successors of successor cardinals.

The following two results from [FHK13], show that, assuming the consistency of a supercompact, model-existence in $\aleph_\beta$ for $\mathcal{L}_{\omega_1,\omega}$-sentences is not absolute between transitive models of ZFC+GCH, for every countable $\beta > \omega$ not of the form $\alpha + 2$.

Theorem 1.2 ([FHK13], Theorem 6)**.** *Assume a ground model V of ZFC+GCH in which there is a supercompact cardinal and an inner model $M \subset V$ of ZFC+GCH and "$\square^*_\lambda$ holds at every singular cardinal $\lambda < \aleph_{\omega_1}$." Then there is a generic extension $V[G]$ in which the GCH is true and model-existence in $\aleph_{\alpha+1}$ for $\mathcal{L}_{\omega_1,\omega}$-sentences is not absolute between M and $V[G]$ for all limit $\alpha < \omega_1^M$.*

Theorem 1.3 ([FHK13], Section 3.5)**.** *Assume M and V is as in Theorem 1.2. Then there is a generic extension $V[G]$ in which the GCH is true and model-existence in $\aleph_\alpha$ for $\mathcal{L}_{\omega_1,\omega}$-sentences is not absolute between M and $V[G]$ for all limit $\alpha < \omega_1^M$ except possibly ω.*

Theorem 1.2 is proved as follows: Given $\alpha < \omega_1^M$, there is an $\mathcal{L}_{\omega_1,\omega}$-sentence $\varphi^{\alpha+1}$ which has a model of size $\aleph_{\alpha+1}$ if and only if there is a special $\aleph_{\alpha+1}$-Aronszajn tree. Moreover, this equivalence is absolute between transitive models of ZFC that contain $\varphi^{\alpha+1}$. Therefore, $\varphi^{\alpha+1}$ has a model of

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[b] The proofs of Theorems 1.1-1.3 obtain $\omega_1^{V[G]} = \omega_1^V$.

[c] In Theorems 1.1-1.3, the restriction $\alpha < \omega_1^M$, as opposed to $\alpha < \omega_1^V$, is not essential. More precisely, we can arrange that $M = V$ by starting from an arbitrary transitive model W of ZFC+GCH and the large cardinal hypothesis and then by a small forcing obtain a generic extension $V = W[H]$ satisfying the hypotheses for M.

size $\aleph_{\alpha+1}$ in M. Moreover, assuming a supercompact, there is a forcing extension $V[G]$ in which GCH is true, but there are no special $\aleph_{\alpha+1}$-Aronszajn trees, for all countable limit α.

Theorem 1.3 has a similar proof, but now for every limit $\omega < \alpha < \omega_1^M$, there exists some $\mathcal{L}_{\omega_1,\omega}$-sentence ψ^α that codes multiple special Aronszajn trees simultaneously. The vocabulary of ψ^α contains predicates Q_β, for all $\beta < \alpha$. Each $Q_{\beta+1}$ is $|Q_\beta|$-like and if $|Q_{\beta+1}|=|Q_\beta|^+$, then ψ^α codes a special $|Q_{\beta+1}|$-Aronszajn tree. It follows that ψ^α has models of size $\aleph_\alpha$ if and only if there are special $\aleph_{\beta+1}$-Aronszajn trees for all $\beta < \alpha$. Moreover, this equivalence is absolute between transitive models of ZFC that contain ψ^α. Thus, ψ^α has models of size $\aleph_\alpha$ in M while in the generic extension $V[G]$ used in the proof of Theorem 1.2, all models of ψ^α have size at most $\aleph_\omega$.

The above argument fails for $\alpha = \omega$ because GCH implies a special $\aleph_{n+1}$-Aronszajn tree for each $n < \omega$. We overcome this barrier by using *coherent* special κ-Aronszajn trees (see Definition 2.1 for coherence).

The following is Corollary 3.15(a) in [Kö03].

Theorem 1.4. *If $\square_\lambda$ holds, then there is a coherent special λ^+-Aronszajn tree.*

So, in L, our modified formula ϕ_α will have a model of size $\aleph_\alpha$. On the other hand, from a model of ϕ_α of size $\geq \aleph_2$, we can recover a coherent pseudotree that contains a cofinal special $\aleph_2$-Aronszajn tree. By coherence, this pseudotree cannot contain a copy of $2^{\leq\omega}$. Todorčević showed that after Levy collapsing a Mahlo to ω_2, every special $\aleph_2$-Aronszajn tree contains a copy of $2^{<\omega_1}$. (We state his corresponding equiconsistency theorem below.) Therefore, there is a model of GCH in which ϕ_α has no models of size $\geq \aleph_2$.

Theorem 1.5 ([Tod81], Theorem 4.6)**.** *Con(ZFC+"there exists a Mahlo")$\leftrightarrow$ Con(ZFC+GCH+"every special $\aleph_2$-Aronszajn tree contains a copy of $2^{<\omega_1}$").*

Our result works not only for $\alpha = \omega$, but for all countable α. For α a limit ordinal, or the successor to a limit ordinal, our result improves the large cardinal assumption from a supercompact (Theorems 1.2, 1.3) to a Mahlo cardinal (Theorem 2.4).

This completes all cases of the absoluteness question for model-existence of $\mathcal{L}_{\omega_1,\omega}$-sentences. The same question can be asked about the amalgamation property of $\mathcal{L}_{\omega_1,\omega}$-sentences. Before we phrase the question precisely, notice that for the amalgamation property we need to specify the type of embeddings used. For this paper we consider only amalgamation under the substructure relation.

Definition 1.6. *Given a collection of models* $\mathbf{K}$*, by the* amalgamation spectrum *of* $\mathbf{K}$*, in symbols AP-spec*($\mathbf{K}$)*, we mean the set of all cardinals κ for which the class of all models in* $\mathbf{K}$ *of size κ is nonempty and has the amalgamation property.*

If $\mathbf{K}$ *is the collection of all models of some sentence φ, then we write AP-spec(φ) for AP-spec*($\mathbf{K}$)*.*

Then the absoluteness question for amalgamation is the following: Is it the case that, if $V \subseteq W$ are transitive models of ZFC with the same ordinals, $\varphi \in V$ and $V \models$ "φ is an $\mathcal{L}_{\omega_1,\omega}$-sentence", then $V \models$ "$\aleph_\alpha \in$ AP-spec(φ)" if and only if $W \models$ "$\aleph_\alpha \in$ AP-spec(φ)"?

Parallel to [FHK13], we can show that manipulating the size of the continuum yields a non-absoluteness result for the amalgamation spectrum of $\mathcal{L}_{\omega_1,\omega}$-sentences.

Consider the sentence ϕ that asserts the existence of a full binary tree of length ω. This sentence has models up to cardinality continuum. All models of ϕ differ only on the maximal branches they contain. In particular, they satisfy the amalgamation property in all cardinals up to the continuum. It follows that the κ-amalgamation property is not absolute for $\kappa \geq \aleph_2$. A similar result, but for $\aleph_2 \leq \kappa \leq 2^{\aleph_1}$, is proved in [SS] using Kurepa trees. The result is interesting mainly when GCH fails, since under GCH, $\aleph_2 = 2^{\aleph_1}$. In [SS], the question was raised about the absoluteness of κ-amalgamation, for $\kappa \geq \aleph_3$, assuming GCH. In Section 3 we answer the question for all $\kappa = \aleph_\alpha$, $3 \leq \alpha < \omega$, and we prove that our examples cannot be used to settle the question for $\alpha \geq \omega$.

2. Model- Existence

In this section we use coherent special Aronszajn trees to prove Theorem 2.4 about non-absoluteness of model-existence. Recall that well-orderings cannot be characterized by an $\mathcal{L}_{\omega_1,\omega}$-sentence. So, it is unavoidable that we will be working with non-well-founded trees. We call such trees *pseudotrees* to distinguish them from their well-founded counterparts.

Definition 2.1.

- *A* pseudotree *is a partial ordered set T such that each strict lower cone $\downarrow x = \{y \mid y <_T x\}$ is a chain.*
- *A pseudotree T is* functional *if there is a linear order L such that T is a downward closed suborder of the class of all functions with domains of the form $\downarrow x = \{y \mid y <_L x\}$, ordered by inclusion. In this case, define a* rank *$\rho\colon T \to L$ by $\rho(t)$ being the unique element such that* $\operatorname{dom}(t) = \downarrow \rho(t)$.
- *Given T and L as above and $x \in L$, define T_x to be the fiber $\rho^{-1}(x)$.*
- *The* cofinality $\operatorname{cf}(T)$ *of a functional pseudotree T is the cofinality* $\operatorname{cf}(\rho[T])$.
- *By $=^*$ we mean equality of sets modulo a finite set.*
- *A pseudotree T is* coherent *if it is functional and* $\operatorname{dom}(s) = \operatorname{dom}(t)$ *implies $s =^* t$.*
- *Given a regular uncountable cardinal κ, a* κ-pseudotree *is a pseudotree T of cofinality κ such that $|T_x| < \kappa$ for each $x \in \rho[T]$.*
- *A κ^+-pseudotree is* special *if it is the union of κ-many of its antichains.*

Lemma 2.2. *If T is a coherent pseudotree T of uncountable cofinality, no suborder of T is isomorphic to $(2^{\leq\omega}, \subset)$.*

Proof. Seeking a contradiction, suppose $e\colon 2^{\leq\omega} \to T$ is an order embedding. Choose $t \in T$ such that $\rho(t) \geq \rho(e(c))$ for all $c \in 2^{<\omega}$. This is possible since T has uncountable cofinality. Then construct $w \in 2^\omega$ as follows. Given $c = w \restriction n$, since $e(c^\frown 0) \perp e(c^\frown 1)$, we may choose $w(n) = i < 2$ such that $e(c^\frown i)(y) \neq t(y)$ for some $y \geq \rho(e(c))$. Thus, $e(w)(y) \neq t(y)$ for infinitely many y, in contradiction with coherence of T. $\square$

Lemma 2.3. *Given $1 \leq \alpha < \omega_1$, there is an $\mathcal{L}_{\omega_1,\omega}$ formula ϕ_α satisfying the following.*

1. *If ϕ_α has a model $\mathfrak{A}$ of size $\geq \aleph_2$, then there is a coherent pseudotree T with cofinality ω_2 and an order embedding of a special $\aleph_2$-Aronszajn tree into T.*
2. *If there is a coherent special $\aleph_{\beta+1}$-Aronszajn tree for each $\beta < \alpha$, then ϕ_α has a model $\mathfrak{B}$ of size $\aleph_\alpha$.*
3. *There is no model of ϕ_α of size greater than $\aleph_\alpha$.*

Proof. Let $1 \leq \alpha < \omega_1$. We will use a predicate symbol ω_β for each $\beta \leq \alpha$, a binary relation symbol $<$, ternary relation symbols L_β and S_β for each $\beta < \alpha$, and a 4-ary relation symbol T_β for each $\beta < \alpha$. Our sentence ϕ_α will assert that the predicates L_β, T_β, and S_β are functional, *i.e.*, each of these predicates defines the graph of a function. Therefore, we will freely write, for example, $z = L_\beta(x, y)$ to denote the unique z such that $L_\beta(x, y, z)$. Further, $L_\beta(x, \bullet)$ will denote the function sending y to $L_\beta(x, y)$.

The idea behind the definition below is that for each $\beta < \alpha$, the relation T_β defines a functional pseudotree with underlying order $\omega_{\beta+1}$. $T_\beta(x, \bullet, \bullet)$ will enumerate the set of all functions in the pseudotree with rank equal to x. Each such function will equal $T_\beta(x, y, \bullet)$, for some $y \in \omega_\beta$. So, each level of the pseudotree will have size at most $|\omega_\beta|$. Also, $\mathrm{dom}(T_\beta(x, y, \bullet)) = \downarrow x$. We will use S_β to witness the fact that T_β is special.

Let $\phi_\alpha \in \mathcal{L}_{\omega_1,\omega}$ assert the following statements for each $\beta < \alpha$, $x \in \omega_{\beta+1}$, and $y \in \omega_\beta$.

(1) The universe is a continuously increasing union $\bigcup_{\beta \leq \alpha} \omega_\beta$ and strictly linearly ordered by $<$.
(2) ω_0 is infinite yet $\downarrow n = \{m \mid m < n\}$ is finite for all $n \in \omega_0$.
(3) ω_β is $<$-downward closed.
(4) L_β, T_β and S_β are functional predicates.
(5) If $\downarrow x$ is not empty, then $L_\beta(x, \bullet)$ is a surjection from ω_β to $\downarrow x$. This will ensure that each $\omega_{\beta+1}$ is $|\omega_\beta|$-like.
(6) $\mathrm{dom}(T_\beta(x, y, \bullet)) = \downarrow x$ and $T_\beta(x, y, \bullet) =^* T_\beta(x, z, \bullet)$, for each $z \in \omega_\beta$, as required for coherence.
(7) For each $w < x$, there exists $z \in \omega_\beta$ such that $T_\beta(w, z, \bullet) \subset T_\beta(x, y, \bullet)$, *i.e.*, T_β is downward closed.
(8) $S_\beta : \omega_{\beta+1} \times \omega_\beta \to \omega_\beta$.
(9) For each $w < x$ and $z \in \omega_\beta$, if $T_\beta(w, z, \bullet) \subset T_\beta(x, y, \bullet)$, then $S_\beta(w, z) \neq S_\beta(x, y)$.

Assuming there exists a model $\mathfrak{A}$ of ϕ_α of size $\geq \aleph_2$, there is a unique $\beta < \alpha$ such that $\left|\omega_\beta^{\mathfrak{A}}\right| = \aleph_1$ and $\mathrm{cf}(\omega_{\beta+1}^{\mathfrak{A}}) = \omega_2$. Why? First, by regularity of $\aleph_2$, the least $\gamma \leq \alpha$ such that $\left|\omega_\gamma^{\mathfrak{A}}\right| \geq \aleph_2$ must be a successor ordinal $\beta + 1$. Second, by (5), $|I| \leq \left|\omega_\beta^{\mathfrak{A}}\right|$ for every proper initial segment I of $\omega_{\beta+1}^{\mathfrak{A}}$. Therefore, $\left|\omega_\beta^{\mathfrak{A}}\right| = \aleph_1$, $\left|\omega_{\beta+1}^{\mathfrak{A}}\right| = \aleph_2$, and $\omega_{\beta+1}^{\mathfrak{A}}$ cannot be covered by $\aleph_1$-many proper initial segments.

Select $W \subset \omega_{\beta+1}^{\mathfrak{A}}$ such that $(W, <) \cong (\omega_2, \in)$, and define trees T and U as follows:

$$T = \left(\left\{T_\beta^{\mathfrak{A}}(x, y, \bullet) \mid (x, y) \in \omega_{\beta+1}^{\mathfrak{A}} \times \omega_\beta^{\mathfrak{A}}\right\}, \subset\right)$$
$$U = \left(\left\{T_\beta^{\mathfrak{A}}(x, y, \bullet) \mid (x, y) \in W \times \omega_\beta^{\mathfrak{A}}\right\}, \subset\right)$$

Then T is a coherent pseudotree of cofinality ω_2, U is an ω_2-tree and suborder of T, and $S_\beta^{\mathfrak{A}}$ witnesses that U is a special $\aleph_2$-Aronszajn tree. This proves part (1).

For part (2), assuming the existence of a coherent special $\aleph_{\beta+1}$-Aronszajn tree $\Upsilon^{(\beta)}$ for each $\beta < \alpha$, let us construct a model $\mathfrak{B}$ of ϕ_α with size $\aleph_\alpha$. Without loss of generality, each $\Upsilon^{(\beta)}$ is a downward closed suborder of $((\omega_{\beta+1})^{<\omega_{\beta+1}}, \subset)$. For each $\beta < \alpha$, let $\Xi_\beta \colon \Upsilon^{(\beta)} \to \omega_\beta$ witness specialness. Let $\mathfrak{B}$ have universe ω_α with $<^{\mathfrak{B}} = \in$ and $\omega_\beta^{\mathfrak{B}} = \omega_\beta$. For each $\beta < \alpha$ and $\gamma < \omega_{\beta+1}$:

(1) If $\gamma \neq 0$, choose a surjection $L_\beta(\gamma, \bullet) \colon \omega_\beta \to \gamma$ and let $L_\beta^{\mathfrak{B}}(\gamma, \bullet) = L_\beta(\gamma, \bullet)$.
(2) Choose a surjection $\Lambda_\beta(\gamma, \bullet) \colon \omega_\beta \to \Upsilon_\gamma^{(\beta)}$, where $\Upsilon_\gamma^{(\beta)}$ is the γ^{th} level of $\Upsilon^{(\beta)}$, and let $T_\beta^{\mathfrak{B}}(\gamma, \delta, \bullet) = \Lambda_\beta(\gamma, \delta)$ for each $\delta < \omega_\beta$.
(3) Let $S_\beta^{\mathfrak{B}}(\gamma, \bullet) = \Xi_\beta(\Lambda_\beta(\gamma, \bullet))$.

It is immediate that $\mathfrak{B}$ is a model of ϕ_α and $\mathfrak{B}$ has size $\aleph_\alpha$, which proves (2). We finish the proof by noticing that (3) follows directly from the definition. $\square$

Notice that ϕ_α is an incomplete sentence.

Theorem 2.4. *For each $2 \leq \alpha < \omega_1$, let ϕ_α be the sentence from Lemma 2.3.*

1. *Given $2 \leq \alpha < \omega_1$, if $\square_{\aleph_\beta}$ holds for all $\beta < \alpha$, then ϕ_α has a model of size $\aleph_\alpha$. In particular, it is consistent with ZFC+GCH that, for all $2 \leq \alpha < \omega_1$, ϕ_α has a model of size $\aleph_\alpha$.*
2. *It is consistent, relative to the existence of a Mahlo cardinal, that there is a model of ZFC+GCH in which, for each $2 \leq \alpha < \omega_1$, all models of ϕ_α have size at most $\aleph_1$.*

Proof. (1) follows by Lemma 2.3, part (2), and Theorem 1.4. (2) follows from Lemma 2.3, part (1), Lemma 2.2 and Theorem 1.5. □

3. Amalgamation

In this section we consider the absoluteness question for the amalgamation spectra of $\mathcal{L}_{\omega_1,\omega}$-sentences. In particular, we investigate the amalgamation spectra of the sentence ϕ_α from Lemma 2.3 under the *substructure* relation. We fix some notation first.

Definition 3.1. *For each $1 \leq \alpha < \omega_1$, let $(\mathbf{K}_\alpha, \subset)$ be the collection of all models of ϕ_α from Theorem 2.4 equipped with the substructure relation.*

Remark 3.2. $(\mathbf{K}_\alpha, \subset)$ *is not quite an abstract elementary class because ϕ_α is not preserved by arbitrary unions of chains. In particular, parts (6) and (7) of the definition of ϕ_α are not preserved by arbitrary unions. However, this can be remedied by adding Skolem functions for parts (6) and (7). That is, for (6) introduce countably many new predicate symbols (C^β_n), each C^β_n of arity $n+3$, and require that $C^\beta_n(x, y, z, \vec{w})$ holds true if and only if $\vec{w}$ is the vector of all elements w such that $T_\beta(x, y, w)$ is different than $T_\beta(x, z, w)$. By coherence there are only finitely many such w's. For (7), introduce a new 4-ary predicate symbol P and require that $P(x, y, w, z)$ holds true if and only if $w < x$ and z is such that $T_\beta(w, z, \bullet) \subset T_\beta(x, y, \bullet)$. Our results hold true even after such a change.*

We prove that if α is finite, then $\mathbf{K}_\alpha$ fails amalgamation in all cardinalities below $\aleph_\alpha$ (Lemma 3.9), but amalgamation in $\aleph_\alpha$ holds trivially because all models of that size, if any, are maximal (Lemma 3.4). By Theorem 2.4, for $\alpha \geq 2$ it is independent of ZFC+GCH whether there are any models in $\mathbf{K}_\alpha$ of size $\aleph_\alpha$. We conclude that it is consistent with ZFC+GCH that the amalgamation spectrum of $\mathbf{K}_\alpha$ for $\alpha \geq 2$ is consistently empty and consistently equal to $\{\aleph_\alpha\}$ (Theorem 3.10).

Lemma 3.3. *For all $\beta < \alpha < \omega_1$ and $A, B \in \mathbf{K}_\alpha$, if $A \subset B$ and $\omega^A_\beta = \omega^B_\beta$, then $(\omega^B_{\beta+1}, <)$ end-extends $(\omega^A_{\beta+1}, <)$.*

Proof. Assume that there exists some $x \in \omega^B_{\beta+1}$ and $y \in \omega^A_{\beta+1}$ with $x < y$. By definition $L_\beta(y, \bullet)$ is a surjection from ω_β to $\downarrow y$. So, there exists some $z \in \omega^B_\beta$ such that $L^B_\beta(y, z) = x$. Since $\omega^A_\beta = \omega^B_\beta$, z also belongs to A, which further implies that $L^A_\beta(y, z) = x$. So, x must be an element of $\omega^A_{\beta+1}$. □

Lemma 3.4. *Assume $1 \leq \alpha < \omega$. All models, if any, in $\mathbf{K}_\alpha$ of size $\aleph_\alpha$ are $\subset$-maximal.*

Proof. First observe that a model A of ϕ_α that has size $\aleph_\alpha$ must satisfy $|\omega^A_\beta| = \aleph_\beta$, for all $\beta \leq \alpha$, while any strict initial segment of ω^A_β must have size $< \aleph_\beta$.

Next, assume that $A \subset B$ and both A, B are of size $\aleph_\alpha$. We prove by induction on $\beta \leq \alpha$, that $\omega^A_\beta = \omega^B_\beta$.

For $\beta = 0$, this follows by the fact that $(\omega_0, <) \cong (\omega, \in)$. For the inductive step, assume that for some $\beta < \alpha$, $\omega^A_\beta = \omega^B_\beta$. By Lemma 3.3, $\omega^B_{\beta+1}$ is an end-extension of $\omega^A_{\beta+1}$. By the above observation, both $\omega^A_{\beta+1}$ and $\omega^B_{\beta+1}$ have size $\aleph_{\beta+1}$ and each strict initial segment has size $< \aleph_{\beta+1}$.

This leads to a contradiction if we assume that $\omega^A_{\beta+1}$ is a strict initial segment of $\omega^B_{\beta+1}$. Thus, it must be the case that $\omega^A_{\beta+1} = \omega^B_{\beta+1}$.

To finish the proof, observe that if $A \subset B$ and A, B agree on all ω_β's, then A, B are equal. $\square$

Next we prove a series of lemmas that lead to Lemma 3.9 where it is proved that $\mathbf{K}_\alpha$ fails amalgamation below $\aleph_\alpha$. These lemmas do not require α to be finite.

Lemma 3.5. *Assume $1 \le \beta < \alpha < \omega_1$ and $M \in \mathbf{K}_\alpha$ such that $|M| \in \{\aleph_0, \aleph_\beta\}$. Then there exists $N \in \mathbf{K}_\beta$ of size $|M|$. If $|M| = \aleph_\beta$, then N can be chosen to be $\subset$-maximal too.*

Proof. If $|M| = \aleph_0$, then let $J = \beta + 1$. Otherwise, let J denote the set of all $\gamma \le \alpha$ with the property that $\left|\omega^M_\delta\right| < \left|\omega^M_\gamma\right|$ for all $\delta < \gamma$. In both cases, there is a unique order isomorphism $g\colon \beta + 1 \to J$. Note that g is a continuous map from $\beta + 1$ to $\alpha + 1$ and g maps successor ordinals to successor ordinals. For each $\gamma < \beta$, choose a bijection f_γ from $\omega^M_{g(\gamma)}$ to $\omega^M_{g(\gamma+1)-1}$. We construct $N \in \mathbf{K}_\beta$ with universe $\omega^M_{g(\beta)}$ by relabeling the γ^{th} pseudotree of M for $\gamma \in \alpha \cap J$ and eliminating the γ^{th} pseudotree of M for $\gamma \in \alpha \setminus J$.

1. $\omega^N_\gamma = \omega^M_{g(\gamma)}$ for all $\gamma \le \beta$.
2. For each $\gamma < \beta$, $x \in \omega^N_{\gamma+1}$, and $y \in \omega^N_\gamma$:
   - (a) Let $L^N_\gamma(x, y) = L^M_{g(\gamma+1)-1}(x, f_\gamma(y))$. This defines a surjection from ω^N_γ to $\downarrow x$.
   - (b) Let $T^N_\gamma(x, y, \bullet) = T^M_{g(\gamma+1)-1}(x, f_\gamma(y), \bullet)$.
   - (c) Let $S^N_\gamma(x, y) = f^{-1}_\gamma(S^M_{g(\gamma+1)-1}(x, f_\gamma(y)))$.

In the case $|M| = \aleph_\beta$, to see that N is $\subset$-maximal, note that $\left|\omega^N_\gamma\right| = \left|\omega^M_{g(\gamma)}\right| = \aleph_\gamma$ for each $\gamma \le \beta$, which in turn implies that each strict initial segment of ω^N_γ has size $< \aleph_\gamma$, for each $\gamma \le \beta$. Therefore, the proof of Lemma 3.4 shows that N is $\subset$-maximal. $\square$

Definition 3.6. *Given $\alpha < \alpha'$, $M \in \mathbf{K}_\alpha$, and $M' \in \mathbf{K}_{\alpha'}$, we say that M'* end-extends *M if M and M' agree on ω_γ for all $\gamma \le \alpha$ and on $L_\gamma, T_\gamma, S_\gamma$ for all $\gamma < \alpha$.*

Lemma 3.7. *Assume $1 \le \alpha < \omega_1$, $M \in \mathbf{K}_\alpha$, and let L be a linear order of size $\le |M|$. Then there is $N \in \mathbf{K}_{\alpha+1}$ of size $|M|$ such that N end-extends M and $\omega^N_{\alpha+1} \setminus \omega^N_\alpha \cong L$.*

Moreover, we may choose N such that, for each linear order L' of size $|M|$ that end-extends L, there exists $N' \in \mathbf{K}_\alpha$ such that $N \subset N'$, $|N| = |N'|$, and $\omega^{N'}_{\alpha+1} \setminus \omega^{N'}_\alpha \cong L'$.

Proof. Given L and L', we will construct N and N' concurrently so that N depends on L but not on L'. Without loss of generality we may assume that L' and ω^M_α are disjoint.

Fix some injection g from $\omega^M_\alpha \cup L$ to ω^M_α so that both the range of g and its complement have size $|M|$. Extend g to an injection f from $\omega^M_\alpha \cup L'$ to ω^M_α. Let 0^M denote $\min(\omega^M_0)$. End-extend M to $N' \in \mathbf{K}_{\alpha+1}$ as follows.

1. $\omega^{N'}_{\alpha+1} = \omega^M_\alpha + L'$.
2. For $0^M \ne x \in \omega^{N'}_{\alpha+1}$, let $L^{N'}_\alpha(x, \bullet)$ be an arbitrary surjection from ω^M_α to $\downarrow x$.
3. For $x \in \omega^{N'}_{\alpha+1}$ and $y \in \omega^{N'}_\alpha$, let $S^{N'}_\alpha(x, y) = f(x)$ and $T^{N'}_\alpha(x, y, z) = 0^M$ for all $z < x$.

If we also stipulate that $\omega^N_{\alpha+1} = \omega^M_\alpha + L$, the above implicitly defines L^N_α, S^N_α, and T^N_α so that they depend on L but not on L'.

$\square$

Corollary 3.8. *Assume $1 \leq \alpha < \alpha' < \omega_1$, $M \in \mathbf{K}_\alpha$, and a sequence of linear orders L_γ for $\alpha \leq \gamma < \alpha'$ each such that $|L_\gamma| \leq |M|$. Then there is $N \in \mathbf{K}_{\alpha'}$ of size $|M|$ that end-extends M and satisfies $\omega^N_{\gamma+1} \setminus \omega^N_\gamma \cong L_\gamma$ for $\alpha \leq \gamma < \alpha'$.*

Moreover, we may choose N such that, if $M' \in \mathbf{K}_\alpha$ satisfies $M \subset M'$ and $|M| = |M'|$, then there exists some end-extension $N' \in \mathbf{K}_{\alpha'}$ of M' that satisfies $|N'| = |M'|$, $N \subset N'$ and $\omega^{N'}_{\gamma+1} \setminus \omega^{N'}_\gamma \cong L_\gamma$ for $\alpha \leq \gamma < \alpha'$.

Proof. Create N by repeatedly end-extending M using Lemma 3.7 at successor stages and unions at limit stages.

To prove the claim about N' we follow the proof of Lemma 3.7. The differences are now that (a) ω^N_α and $\omega^{N'}_\alpha$ may not be the same and (b) the construction of N and N' guarantees that $\omega^{N'}_{\alpha'} \setminus \omega^N_{\alpha'}$ equals $\omega^{M'}_\alpha \setminus \omega^M_\alpha$, i.e. no new points are added to N other than the points added to M.

The proof is by induction on γ. The limit stages are trivial, so we describe only how L_γ, S_γ and T_γ are defined on the successor stages.

- For $0^M \neq x \in \omega^{N'}_{\gamma+1}$, let $L^{N'}_\gamma(x, \bullet)$ equal $L^N_\gamma(x, \bullet)$ when restricted to domain ω^N_γ, and let $L^{N'}_\gamma(x, \bullet)$ be the identity otherwise.
- Similarly, for $x \in \omega^{N'}_{\gamma+1}$, let $S^{N'}_\gamma(x, \bullet)$ equal $S^N_\gamma(x, \bullet)$ when restricted to domain ω^N_γ, and let $S^{N'}_\gamma(x, \bullet)$ be the identity otherwise.
- For $x \in \omega^{N'}_{\gamma+1}$ and $y \in \omega^{N'}_\gamma$, let $T^{N'}_\gamma(x, y, z) = 0^M$ for all $z < x$.

□

Lemma 3.9. *Let $1 \leq \beta < \alpha < \omega_1$ and $\gamma \in \{0, \beta\}$ and assume that $\mathbf{K}_\alpha$ has a model of size $\aleph_\gamma$. Then amalgamation fails in $\aleph_\gamma$.*

Proof. We give an example of a triple (A, B, C) in $\mathbf{K}_\alpha$ that can not be amalgamated. The reason that amalgamation fails is that linear orders fail amalgamation under end-extension.

Assume $M \in \mathbf{K}_\alpha$ and $|M| = \aleph_\gamma$. The proof splits into two cases: $\gamma = \beta > 0$ and $\gamma = 0$. We give the details for the first case and sketch the proof of the second case.

By Lemma 3.5, there exists a $\subset$-maximal $N \in \mathbf{K}_\gamma$ with $|N| = \aleph_\gamma$. End-extend N using Lemma 3.7 to create three models $A', B', C' \in \mathbf{K}_{\gamma+1}$ that satisfy the following:

(1) $A' \subset B'$ and $A' \subset C'$
(2) $|A'| = |B'| = |C'| = \aleph_\gamma$
(3) $\omega^{A'}_\gamma = \omega^{B'}_\gamma = \omega^{C'}_\gamma = \omega^N_\gamma$
(4) $\omega^{A'}_{\gamma+1} = \omega^{A'}_\gamma + \omega$
(5) $\omega^{B'}_{\gamma+1} = \omega^{A'}_\gamma + \omega \cdot 2$
(6) $\omega^{C'}_{\gamma+1} = \omega^{A'}_\gamma + \omega + \mathbb{Q}$

Then use Corollary 3.8 to end-extend A', B', C' to some $A, B, C \in \mathbf{K}_\alpha$ such that $A \subset B$ and $A \subset C$. Assume D is an amalgam of B and C over A. Then $\omega^D_\gamma = \omega^N_\gamma$ by maximality of N. By Lemma 3.3, $\omega^D_{\gamma+1}$ must be an end-extension of both $\omega^B_{\gamma+1}$ and $\omega^C_{\gamma+1}$. But this is impossible.

For the case when $\gamma = 0$ construct three models $A, B, C \in \mathbf{K}_\alpha$ with $\omega^A_1 = \omega \cdot 2$, $\omega^B_1 = \omega \cdot 3$ and $\omega^C_1 = \omega \cdot 2 + \mathbb{Q}$. The same argument proves that they can not be amalgamated in $\mathbf{K}_\alpha$. □

Theorem 3.10. *Assume $1 \leq \alpha < \omega$. The amalgamation spectrum of $\mathbf{K}_\alpha$ is equal to $\{\aleph_\alpha\}$, if there are models in $\mathbf{K}_\alpha$ of size $\aleph_\alpha$. Otherwise it is empty.*

Proof. First recall that by 2.3(3), $\mathbf{K}_\alpha$ has no models of size greater than $\aleph_\alpha$. By Lemma 3.9 amalgamation fails for all cardinals below $\aleph_\alpha$. If there are models in $\mathbf{K}_\alpha$ of size $\aleph_\alpha$, then $\aleph_\alpha$-amalgamation holds trivially by Lemma 3.4. In this case the amalgamation spectrum is equal to $\{\aleph_\alpha\}$. Otherwise, the amalgamation spectrum is empty. $\square$

Corollary 3.11. *The amalgamation spectrum of* $\mathbf{K}_1$ *is* $\{\aleph_1\}$.

Proof. The existence of coherent special $\aleph_1$-Aronszajn tree follows from Theorem 1.4, because $\square_{\aleph_0}$ holds trivially. $\square$

Theorem 3.12. *The following statements are* not absolute *for transitive models of ZFC.*

(a) The amalgamation spectrum of an $\mathcal{L}_{\omega_1,\omega}$*-sentence is empty.*

(b) For finite $n \geq 2$ *and* ϕ *an* $\mathcal{L}_{\omega_1,\omega}$*-sentence,* $\aleph_n$ *belongs to the amalgamation spectrum of* ϕ.

The results remain true even if we consider transitive models of ZFC+GCH.

Proof. By Theorem 3.10 and Theorem 2.4. $\square$

A couple of notes: Theorem 3.12 covers all cardinals $\aleph_n$, with n finite and $n \geq 2$. For $n = 0$, $\aleph_0$-amalgamation is absolute by an easy application of Shoenfield's absoluteness. The question for $n = 1$ remains open.

Lastly we prove that our examples can not be used to resolve the absoluteness question of $\aleph_\alpha$-amalgamation, for $\omega \leq \alpha < \omega_1$, under GCH. The reason is that in this case $\mathbf{K}_\alpha$ has empty amalgamation spectrum.

Lemma 3.13. *Assume* $\omega \leq \alpha < \omega_1$ *and* $M \in \mathbf{K}_\alpha$. *Let* K *be a countable linear order. Then there is a model* R *in* $\mathbf{K}_\alpha$ *of size* $|M|$ *such that* $\omega_1^R \setminus \omega_0^R \cong K$.

Moreover, we may choose R *such that, for each countable linear order* J *that end-extends* K*, there exists* $N \in \mathbf{K}_\alpha$ *such that* $R \subset N$, $|N| = |R|$, *and* $\omega_1^N \setminus \omega_0^N \cong J$.

Proof. Given K and J, we will construct R and N in parallel, taking care that R depends on K but not on J. The idea is that we move all the pseudotrees of M one level higher and introduce a new pseudotree at the bottom.

For each $\beta \leq \alpha$, define $\sigma(\beta)$ to be $\beta - 1$ if $0 < \beta < \omega$ and β otherwise. Without loss of generality, assume $(\omega_0^M, <^M) = (\omega, \in)$; then define $\omega_0^N = \omega$ and $\omega_\beta^N = \omega + J + (\omega_{\sigma(\beta)}^M \setminus \omega)$ for all $\beta > 0$. In particular, $\omega_1^N = \omega + J$ and $\omega_2^N = \omega + J + (\omega_1^M \setminus \omega)$. For all $\beta \leq \alpha$, let $\omega_\beta^R = \omega_\beta^N \setminus (J \setminus K)$.

Next, we define the bottom pseudotree of N and, implicitly, the bottom pseudotree of R. Choose an injection $g \colon \omega + K \to \omega$ with co-infinite range and then extend it to an injection $f \colon \omega + J \to \omega$. For each $x \in \omega + J$ and $y \in \omega$, declare that:

- $L_0^N(x, \bullet)$ is an arbitrary surjection from ω to $\downarrow^N x$ if $x \neq 0$.
- $T_0^N(x, y, z) = 0$ for all $z <^N x$.
- $S_0^N(x, y) = f(x)$.

The above implicitly defines L_0^R, S_0^R, and T_0^R so that they depend on K but not on J.

Given $1 \leq \beta < \alpha$, observe that $\omega_\beta^N = \omega_{\sigma(\beta)}^M \cup J$ and $\omega_{\beta+1}^N = \omega_{\sigma(\beta)+1}^M \cup J$. Then declare the following for each $x \in \omega_{\beta+1}^N \setminus \{0\}$ and $y \in \omega_\beta^N$.

$$L_\beta^N(x, y) = \begin{cases} L_{\sigma(\beta)}^M(x, y) & : x, y \notin J \\ y & : x >^N y \in J \\ 0 & : x \leq^N y \in J \end{cases}$$

The above defines our needed surjections $L^N_\beta(x,\bullet)$ and $L^R_\beta(x,\bullet)$, except in the case where $x \in J$ and $y \in \omega^M_{\sigma(\beta)}$. To define L^N_β in this case, fix some surjection f_β from $\omega^M_{\sigma(\beta)}$ to ω and let $L^N_\beta(x,y) = f_\beta(y)$.

This completes the definition of $L^N_\beta(x,\bullet)$ and $L^R_\beta(x,\bullet)$, and notice that the latter depends on K but not on J.

Next, we begin defining the β^{th} pseudotree of N by splicing constant functions into the $\sigma(\beta)^{\text{th}}$ pseudotree of M. For each $x \in \omega^M_{\sigma(\beta)+1}$ and $y \in \omega^M_{\sigma(\beta)}$, let

$$T^N_\beta(x,y,z) = \begin{cases} T^M_{\sigma(\beta)}(x,y,z) & : x >^N z \notin J \\ 0 & : x >^N z \in J \end{cases}.$$

The parts of the pseudotrees T^N_β and T^R_β defined so far are coherent. We now fill in the missing levels indexed by J so as to also achieve downward closure. Fix $x_0 \in \omega^M_1 \setminus \omega$ and let (g_β, h_β) map $\omega^M_{\sigma(\beta)}$ onto the set of pairs

$$\left\{(x,y) \mid x_0 \geq^M x \notin \omega \text{ and } y \in \omega^M_{\sigma(\beta)}\right\}.$$

Then declare that, for each $x \in J$ and $y \in \omega^M_{\sigma(\beta)}$,

$$T^N_\beta(x,y,z) = \begin{cases} T^M_{\sigma(\beta)}(g_\beta(y), h_\beta(y), z) & : z \in \omega \\ 0 & : x >^N z \in J \end{cases}.$$

Observe that so far T^R_β depends on K but not on J.

To witness specialness, declare that, for each $x \in \omega^N_{\beta+1}$ and $y \in \omega^M_{\sigma(\beta)}$,

$$S^N_\beta(x,y) = \begin{cases} S^M_{\sigma(\beta)}(x,y) & : x \notin J \\ x & : x \in J \end{cases}.$$

Finally, for each $x \in \omega^M_{\sigma(\beta)+1}$ and $y \in J$, let $T^N_\beta(x,y,\bullet) = T^N_\beta(x,0,\bullet)$ and $S^N_\beta(x,y) = S^N_\beta(x,0)$. This completes the construction of N and R. We have also implicitly defined T^R_β and S^R_β so that they depend on K but not on J. □

Theorem 3.14. *Assume $\omega \leq \alpha < \omega_1$. The amalgamation spectrum of $\mathbf{K}_\alpha$ is empty.*

Proof. By Lemma 3.9, $\mathbf{K}_\alpha$ fails amalgamation in $\aleph_\beta$, for all $\beta < \alpha$. We prove that this is the case even for $\beta = \alpha$. If $\mathbf{K}_\alpha$ has no models of size $\aleph_\alpha$, the result is trivial. So, assume that is a model M of size $\aleph_\alpha$.

We use the same method as for Lemma 3.9. In particular, we construct a triple (A, B, C) with the following properties:

(1) $A \subset B$ and $A \subset C$
(2) $|A| = |B| = |C| = \aleph_\alpha$
(3) $\omega^A_1 = \omega \cdot 2$
(4) $\omega^B_1 = \omega \cdot 3$
(5) $\omega^C_1 = \omega \cdot 2 + \mathbb{Q}$

This is possible by applying Lemma 3.13 twice; once for the pair $\omega \cdot 2 \subset \omega \cdot 3$ and a second time for the pair $\omega \cdot 2 \subset \omega \cdot 2 + \mathbb{Q}$.

As in Lemma 3.9, if D were an amalgam of B and C over A, then ω^D_1 would must be an end-extension of both ω^B_1 and ω^C_1, which is impossible. □

4. Open Problems

The following are some questions that remain open. Some of the questions do not bear much resemblance to the results of this paper. Nevertheless we encountered these questions during our search for a proof to Theorem 2.4.

(1) Can the consistency strength of our non-absoluteness theorem be further reduced? In particular, is it possible to prove the same result without any large cardinal assumptions?
(2) If α is countably infinite, is $\aleph_\alpha$-amalgamation non-absolute for transitive models of ZFC+GCH?
(3) Is $\aleph_1$-amalgamation for $\mathcal{L}_{\omega_1,\omega}$-sentences absolute for transitive models of ZFC?
(4) The way we proved non-absoluteness of amalgamation in $\aleph_n$, for finite n, is by choosing appropriate set-theoretic assumptions that affect the model-existence spectrum. If there are no models in $\aleph_n$ then the amalgamation question becomes void. Can we prove non-absoluteness of amalgamation in $\aleph_n$ in the following stronger form: There are two transitive models of ZFC, say $V \subset W$, with the same ordinals, a sentence ϕ that belongs to $(\mathcal{L}_{\omega_1,\omega})^V$, both V and W satisfy "ϕ has a model of size $\aleph_n$", and V, W disagree on "models of ϕ of size $\aleph_n$ satisfy amalgamation"? Same question is open for $\aleph_n$-joint embedding.
(5) The principle $\square_\kappa$ asserts the existence of a square sequence, i.e. a sequence $< C_\alpha | \alpha \in Lim(\kappa^+) >$ that satisfies (i) C_α is a club of α, (ii) if $cf(\alpha) < \kappa$, then $|C_a| < \kappa$ and (iii) if $\beta \in Lim(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$. Are there any κ^+-like linear orders $(L, <)$ (other than well-orders) for which the existence of a sequence $< C_\alpha | \alpha \in Lim(L) >$ that satisfies (i)-(iii) is independent of ZFC?
(6) The proof of Lemma 2.3 does not quite recover a coherent special $\aleph_2$-Aronszajn tree from an $\aleph_\alpha$-sized model of ϕ_α. It merely recovers a special $\aleph_2$-Aronszajn tree that embeds in a coherent pseudotree. Is there an $\mathcal{L}_{\omega_1,\omega}$-sentence for which existence of a model of size $\aleph_2$ entails a coherent special $\aleph_2$-Aronszajn tree?
(7) One strategy for reducing our large cardinal assumption from Mahlo to inaccessible is to attempt to code Kurepa trees using formulas satisfied by higher-gap simplified morasses. The following test question captures the core obstacle to this strategy. Assume $V = L$. Choose $\mathfrak{M} = (L(\delta), \in)$ such that $\omega_4 < \delta < \omega_5$ and $\mathfrak{M} \prec (L(\omega_5), \in)$. Let G be a generic filter of the Miller-like version of Namba forcing. (This forcing is the Nm defined in XI.4.1 of [Sh]; the Laver-like version of Namba forcing is the Nm$'$ defined in the remark after XI.4.1A.) Then, in $V[G]$, GCH and the regularity of ω_1^V and ω_4^V are preserved but $\mathrm{cf}(\omega_2^V)$ collapses to ω and $\mathrm{cf}(\omega_3^V)$ collapses to ω_1. In V, let ψ be an $\mathcal{L}_{\omega_1,\omega}$ formula that defines a binary relation on the structure $\mathfrak{M}$, possibly using parameters from $\mathfrak{M}$. Can ψ be chosen independently of G such that in $V[G]$ we have $\left|\mathrm{dom}\left(\psi^{\mathfrak{M}}\right)\right| = \aleph_2$, $\left|\mathrm{ran}\left(\psi^{\mathfrak{M}}\right)\right| = \aleph_1$, and $\left\{\psi^{\mathfrak{M}}[x] \cap y \mid x \in \mathrm{dom}\left(\psi^{\mathfrak{M}}\right)\right\}$ countable for all countable sets y?
(8) In the above test question, we specified "Miller-like" because we can prove, assuming CH, that this version of Namba forcing does not add cofinal branches to ω_1-trees in the ground model, thus opening the door to rcs iterated forcing extensions without any Kurepa trees (assuming an inaccessible in the ground model). However, our proof's fusion argument works for Miller and Sacks but not Laver type tree forcings. This leads us to ask, is it consistent with CH that Laver forcing adds a cofinal branch to some ω_1-tree in the ground model?

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