ON A THEOREM OF VAN MILL

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ABSTRACT. We fix an error in the proof of a theorem of Van Mill about homeomorphisms between compactifications of ω with small weight.

Theorem 4.2. *Let $a\omega$ and $b\omega$ be compactifications of ω. Assume that*

(1) *there is a retraction $r\colon a\omega \to a\omega \setminus \omega$,*
(2) *there is a retraction $s\colon b\omega \to b\omega \setminus \omega$,*
(3) *$f\colon a\omega \setminus \omega \to b\omega \setminus \omega$ is a homeomorphism.*

If the weight of $a\omega \setminus \omega$ is less than $\mathfrak{p}$, then f can be extended to a homeomorphism $\bar{f}\colon a\omega \to b\omega$.

The above is Van Mill's Theorem 4.2 from [5]. Andrea Medini noticed an error in Van Mill's proof. In short, Claim 3 of that proof is wrong, though plausible at first reading. Upon close inspection, it is seen that while

$$\pi[r^{-1}[U_{\alpha,i}] \cap M_1] \subseteq^* s^{-1}[V_{\alpha,i}] \cap N_1 \text{ and } \pi[r^{-1}[U_{\alpha,i}] \cap M_2] \supseteq^* s^{-1}[V_{\alpha,i}] \cap N_2$$

are true, neither $\pi[r^{-1}[U_{\alpha,i}] \cap \omega] \subseteq^* s^{-1}[V_{\alpha,i}]$ nor $\pi[r^{-1}[U_{\alpha,i}] \cap \omega] \supseteq^* s^{-1}[V_{\alpha,i}]$ is true in general.

I propose the following proof of the theorem.

Lemma 1. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are boolean subalgebras of $\mathcal{P}(\omega)$ such that both $\mathcal{A}$ and $\mathcal{B}$ has size less than $\mathfrak{p}$ and the empty set is the only finite set in $\mathcal{A} \cup \mathcal{B}$. If H is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, then there is a permutation g of ω such that $g[A] =^* H(A)$ for all $A \in \mathcal{A}$.*

Proof. Fix H as above. By Bell's Theorem [1], it suffices to exhibit a σ-centered forcing $\mathbb{P}$ and a set $\mathcal{D}$ of fewer than $\mathfrak{p}$-many dense subsets of $\mathbb{P}$ such that a map g as above can be constructed from an arbitrary filter G of $\mathbb{P}$ that meets every set in $\mathcal{D}$.

Let $\mathbb{P}$ be a forcing order with conditions of the form $p = \langle g_p, \mathcal{F}_p \rangle$ where g_p is an injective finite partial function from ω to ω and $\mathcal{F}_p$ is a finite subalgebra of $\mathcal{A}$. Order $\mathbb{P}$ by declaring $q \le p$ if $g_q \supseteq g_p$, $\mathcal{F}_q \supseteq \mathcal{F}_p$, and $(g_q \setminus g_p)[U] \subseteq H(U)$ for all $U \in \mathcal{F}_p$. If $p, q \in \mathbb{P}$ and $g_p = g_q$, then $\langle g_p, \mathcal{E} \rangle$ is a common extension of p and q if $\mathcal{E}$ is a finite subalgebra of $\mathcal{A}$ such that $\mathcal{E} \supseteq \mathcal{F}_p \cup \mathcal{F}_q$. Thus, $\mathbb{P}$ is σ-centered. Moreover, we could have chosen $\mathcal{E}$ to contain an arbitrary element of $\mathcal{A}$, so the set $D_A = \{p \in \mathbb{P} : A \in \mathcal{F}_p\}$ is dense for all $A \in \mathcal{A}$.

For each $n < \omega$, let D'_n and D''_n respectively denote $\{p \in \mathbb{P} : n \in \mathrm{dom}(g_p)\}$ and $\{p \in \mathbb{P} : n \in \mathrm{ran}(g_p)\}$. Given $n < \omega$ and $p \in \mathbb{P} \setminus D'_n$, let A be the unique atomic

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element of $\mathcal{F}_p$ such that $n \in A$. For all of the infinitely many $m \in H(A) \setminus \operatorname{ran}(g_p)$, we have $\langle g_p \cup \{\langle n, m\rangle\}, \mathcal{F}_p\rangle \leq p$, so D'_n is dense. By symmetry, D''_n is also dense.

By Bell's Theorem, there is a filter G of $\mathbb{P}$ such that G meets D_A, D'_n, and D''_n for all $A \in \mathcal{A}$ and $n < \omega$. Set $g = \bigcup\{g_p : p \in G\}$. Then g is a permutation of ω. Fix $A \in \mathcal{A}$ and choose $p \in G \cap D_A$. Since G is a filter of $\mathbb{P}$, we have $(g \setminus g_p)[A] \subseteq H(A)$ and $(g \setminus g_p)[\omega \setminus A] \subseteq H(\omega \setminus A) = \omega \setminus H(A)$. Hence, $g[A] =^* H(A)$ as desired. □

Question 2. Is the following strengthening of Lemma 1 true?

> Suppose $\mathcal{A}$ and $\mathcal{B}$ are boolean subalgebras of $\mathcal{P}(\omega)$ such that both $\mathcal{A}$ and $\mathcal{B}$ has size less than $\mathfrak{p}$. If H is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ such that $H\left[\mathcal{A} \cap [\omega]^{<\omega}\right] = \mathcal{B} \cap [\omega]^{<\omega}$, then there is a permutation g of ω such that $g[A] =^* H(A)$ for all $A \in \mathcal{A}$.

Definition 3. Given a space X, let $\mathrm{RO}(X)$ denote the boolean algebra of regular open subsets of X.

Proof of Theorem 4.2. Set $\kappa = w(a\omega \setminus \omega)$. Let $\mathcal{W}$ be a base of $a\omega \setminus \omega$ consisting of κ-many regular open sets each with boundary disjoint from the countable set $r[\omega] \cup f^{-1}[s[\omega]]$. Let $\mathcal{U}$ be the boolean subalgebra of $\mathrm{RO}(a\omega \setminus \omega)$ generated by $\mathcal{W}$. Set $\mathcal{U}' = \{r^{-1}[U] : U \in \mathcal{U}\}$.

Claim. *Because r is a retraction and ω is a discrete open subset of $a\omega$, the r-preimages of regular open sets are also regular open. Moreover, since all elements of $\mathcal{U}$ have boundary avoiding $r[\omega]$, the set $\mathcal{U}'$ is a subalgebra of $RO(a\omega)$ that is isomorphic to $\mathcal{U}$.*

Deferring the proof of the claim for now, set $\mathcal{A} = \{\omega \cap U : U \in \mathcal{U}'\}$. Since ω is dense in $a\omega$, $\mathcal{A}$ is a subalgebra of $\mathrm{RO}(\omega)$ that is isomorphic to $\mathcal{U}'$. Since ω is discrete, $\mathrm{RO}(\omega) = \mathcal{P}(\omega)$. Thus, $\mathcal{A}$ is a subalgebra of $\mathcal{P}(\omega)$ that is isomorphic to $\mathcal{U}$.

Set $\mathcal{V} = \{f[U] : U \in \mathcal{U}\}$, $\mathcal{V}' = \{s^{-1}[V] : V \in \mathcal{V}\}$, and $\mathcal{B} = \{\omega \cap V : V \in \mathcal{V}'\}$. By symmetry, $\mathcal{B}$ is a subalgebra of $\mathcal{P}(\omega)$ that is isomorphic to $\mathcal{V}$. Since f is a homeomorphism, $\mathcal{A}$ and $\mathcal{B}$ are isomorphic subalgebras of $\mathcal{P}(\omega)$ each with size less than $\mathfrak{p}$. The isomorphism is defined by $\omega \cap r^{-1}[U] \mapsto \omega \cap s^{-1}[f[U]]$ for all $U \in \mathcal{U}$.

Let us show that $\mathcal{A} \cap [\omega]^{<\omega} = \{\emptyset\}$. Suppose $U \in \mathcal{U}$ and $\omega \cap r^{-1}[U]$ is finite. This implies U has empty interior in $a\omega \setminus \omega$, which implies $U = \emptyset$ because U is regular open. Thus, $\mathcal{A} \cap [\omega]^{<\omega} = \{\emptyset\}$ as desired. By symmetry, $\mathcal{B} \cap [\omega]^{<\omega} = \{\emptyset\}$.

By Lemma 1, there exists a permutation g of ω such that $g[\omega \cap r^{-1}[W]] \subseteq^* \omega \cap s^{-1}[f[W]]$ for all $W \in \mathcal{W}$. Set $\bar{f} = f \cup g$. Let us show that $\bar{f}$ is our desired homeomorphism from $a\omega$ to $b\omega$. By compactness, is suffices to show that $\bar{f}$ is continuous. Let Z be an open subset of $b\omega$ and let $x \in \bar{f}^{-1}[Z]$. It suffices to show that x is in the interior of $\bar{f}^{-1}[Z]$. We may assume without loss of generality that $x \notin \omega$. Choose $W \in \mathcal{W}$ such that $x \in W \subseteq \overline{W} \subseteq f^{-1}[Z \setminus \omega]$. It suffices to show that there is a finite $\sigma \subseteq \omega$ such that $r^{-1}[W] \setminus \sigma \subseteq \bar{f}^{-1}[Z]$. Equivalently, we need only show that $r^{-1}[W] \setminus \bar{f}^{-1}[Z]$ is finite, which is true if and only if $\bar{f}[r^{-1}[W]] \setminus Z$ is finite. By the definition of $\bar{f}$, we have

$$\bar{f}[r^{-1}[W]] \setminus Z = (g[\omega \cap r^{-1}[W]] \cup f[W]) \setminus Z = g[\omega \cap r^{-1}[W]] \setminus Z \subseteq^* \omega \cap s^{-1}[f[W]] \setminus Z,$$

so it suffices to show that $s^{-1}[f[W]] \setminus Z$ is finite. This set is indeed finite because it is contained in $s^{-1}\left[\overline{f[W]}\right] \setminus Z$, which is a compact subset of ω. □

Proof of Claim. Set $X = a\omega \setminus \omega$ and $Y = a\omega$. Fix $R \in \mathrm{RO}(X)$; let us show that $r^{-1}[R] \in \mathrm{RO}(Y)$. Fix $p \in \mathrm{int}_Y \,\mathrm{cl}_Y\, r^{-1}[R]$. It suffices to show that $p \in r^{-1}[R]$. Since ω is a discrete open subset of Y, we may assume $p \in X$. Choose a Y-neighborhood Z of p such that $Z \subseteq \mathrm{cl}_Y\, r^{-1}[R]$. The set $Z \cap X$ is an X-neighborhood of p and we have

$$Z \cap X \subseteq X \cap \mathrm{cl}_Y\, r^{-1}[R] \subseteq X \cap r^{-1}[\mathrm{cl}_X\, R] = \mathrm{cl}_X\, R.$$

Hence, $r(p) = p \in \mathrm{int}_X \,\mathrm{cl}_X\, R = R$, so $p \in r^{-1}[R]$.

Now fix $U, V \in \mathcal{U}$. To show that $\mathcal{U}'$ is a subalgebra of $\mathrm{RO}(Y)$ isomorphic to $\mathcal{U}$, it suffices to prove that $r^{-1}[U \cap V] = r^{-1}[U] \cap r^{-1}[V]$ and

$$r^{-1}[\mathrm{int}_X \,\mathrm{cl}_X(U \cup V)] = \mathrm{int}_Y \,\mathrm{cl}_Y(r^{-1}[U] \cup r^{-1}[V]).$$

The former equation is trivially true; for the latter equation, fix $p \in \mathrm{int}_Y \,\mathrm{cl}_Y(r^{-1}[U] \cup r^{-1}[V])$ and $q \in r^{-1}[\mathrm{int}_X \,\mathrm{cl}_X(U \cup V)]$. It suffices to show that $p \in r^{-1}[\mathrm{int}_X \,\mathrm{cl}_X(U \cup V)]$ and $q \in \mathrm{int}_Y \,\mathrm{cl}_Y(r^{-1}[U] \cup r^{-1}[V])$.

Start with p. Trivially, $p \in \mathrm{int}_Y \,\mathrm{cl}_Y\, r^{-1}[U \cup V]$; hence, p has a Y-neighborhood Z that is contained in $\mathrm{cl}_Y\, r^{-1}[U \cup V]$, which is contained in $r^{-1}[\mathrm{cl}_X(U \cup V)]$. If $p \in X$, then

$$r(p) = p \in Z \cap X \subseteq \mathrm{int}_X \left(X \cap r^{-1}[\mathrm{cl}_X(U \cup V)]\right) = \mathrm{int}_X \,\mathrm{cl}_X(U \cup V),$$

so $p \in r^{-1}[\mathrm{int}_X \,\mathrm{cl}_X(U \cup V)]$ as desired. If $p \in \omega$, then p is in $r^{-1}[U \cup V]$ because p is isolated in Y, so $p \in r^{-1}[\mathrm{int}_X \,\mathrm{cl}_X(U \cup V)]$ as desired.

Now consider q. If $q \in \omega$, then $q \in r^{-1}[U \cup V]$ because the boundaries of U and V avoid $r[\omega]$. In this case, $q \in \mathrm{int}_Y \,\mathrm{cl}_Y(r^{-1}[U] \cup r^{-1}[V])$ follows from continuity of r. Therefore, we may assume $q \in X$. Hence, $q \in \mathrm{int}_X \,\mathrm{cl}_X(U \cup V)$, so q has an X-neighborhood Z contained in $\mathrm{cl}_X(U \cup V)$. Hence, $r^{-1}[Z]$ is a Y-neighborhood of q. Therefore, it suffices to show that $r^{-1}[Z]$ is contained in $\mathrm{cl}_Y\, r^{-1}[U \cup V]$. Fix $z \in r^{-1}[Z]$; it suffices to show that $z \in \mathrm{cl}_Y\, r^{-1}[U \cup V]$. If $z \in X$, then

$$z \in Z \subseteq \mathrm{cl}_X(U \cup V) \subseteq \mathrm{cl}_Y(U \cup V) \subseteq \mathrm{cl}_Y\, r^{-1}[U \cup V].$$

Hence, we may assume $z \in \omega$. Therefore,

$$r(z) \in r[\omega] \cap Z \subseteq r[\omega] \cap \mathrm{cl}_X(U \cup V) \subseteq U \cup V.$$

Hence, $z \in r^{-1}[U \cup V] \subseteq \mathrm{cl}_Y\, r^{-1}[U \cup V]$. □

Van Mill's Theorem 4.2 is directly cited in [2] and indirectly used in [3] and [4]. However, none of these papers use Van Mill's proof of Theorem 4.2. Therefore, changing the proof of Theorem 4.2 does not make it necessary to change any of these papers.

References

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