NOETHERIAN TYPES OF HOMOGENEOUS COMPACTA AND DYADIC COMPACTA

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ABSTRACT. The Noetherian type of a space is the least κ such that it has a base that is κ-like with respect to reverse inclusion. Just as all known homogeneous compacta have cellularity at most $\mathfrak{c}$, they satisfy similar upper bounds in terms of Noetherian type and related cardinal functions. We prove these and many other results about these cardinal functions. For example, every homogeneous dyadic compactum has Noetherian type ω. Assuming GCH, every point in a homogeneous compactum X has a local base that is $c(X)$-like with respect to containment. If every point in a compactum has a well-quasiordered local base, then some point has a countable local π-base.

1. INTRODUCTION

Van Douwen's Problem (see Kunen [16]) asks whether there is a homogeneous compactum of cellularity exceeding $\mathfrak{c}$. (See Engelking [7], Juhász [14], and Kunen [17] for all undefined terms. In particular, recall that $w(\cdot)$, $\pi(\cdot)$, $\chi(\cdot)$, $\pi\chi(\cdot)$, $d(\cdot)$, $c(\cdot)$, and $t(\cdot)$ respectively denote weight, π-weight, character, π-character, density, cellularity, and tightness of topological spaces.) A homogeneous compactum of cellularity $\mathfrak{c}$ exists by Maurice [19], but van Douwen's Problem remains open in all models of ZFC.

Definition 1.1. We say that a homogeneous compactum is *exceptional* if it is not homeomorphic to a product of dyadic compacta and first countable compacta.

By Arhangel′skiĭ's Theorem, first countable compacta have size at most $\mathfrak{c}$; dyadic compacta are ccc. Since the cellularity of a product space equals the supremum of the cellularities of its finite subproducts (see p. 107 of [14]), all nonexceptional homogeneous compacta have cellularity at most $\mathfrak{c}$. To the best of the author's knowledge, there are only two classes of examples of exceptional homogeneous compacta (see [20]); these two kinds of spaces have cellularities ω and $\mathfrak{c}$.

We investigate several cardinal functions defined in terms of order-theoretic base properties. Just like cellularity, these functions have upper bounds when restricted to the class of known homogeneous compacta. Moreover, GCH implies that one of these functions is a lower bound on cellularity when restricted to homogeneous compacta.

Definition 1.2. Given a cardinal κ, define a poset to be *κ-like* (*κ^{op}-like*) if no element is above (below) κ-many elements. Define a poset to be *almost κ^{op}-like* if it has a κ^{op}-like dense subset.

In the context of families of subsets of a topological space, we will always implicitly order by inclusion.

Definition 1.3. Given a space X, let the *Noetherian type* of X, or $Nt(X)$, be the least $\kappa \geq \omega$ such that X has a base that is κ^{op}-like. Analogously define *Noetherian π-type* in terms of π-bases and denote it by $\pi Nt(X)$. Given a subset E of X, let the *local Noetherian type* of E in X, or $\chi Nt(E, X)$, be the least $\kappa \geq \omega$ such that there is a κ^{op}-like neighborhood base of E. Given $p \in X$, let the local Noetherian type of p, or $\chi Nt(p, X)$, be $\chi Nt(\{p\}, X)$. Let the local Noetherian type of X, or $\chi Nt(X)$, be the supremum of the local Noetherian types of its points. Let the *compact Noetherian type* of X, or $\chi_K Nt(X)$, be the supremum of the local Noetherian types of its compact subsets. We call Nt, πNt, χNt, and $\chi_K Nt$ *Noetherian cardinal functions*.

Noetherian type and Noetherian π-type were introduced by Peregudov [23]. Preceding this introduction are several papers by Peregudov, Šapirovskiĭ and Malykhin [18, 21, 22, 24] about $\min\{Nt(\cdot), \omega_2\}$ and $\min\{\pi Nt(\cdot), \omega_2\}$ (using different terminologies). Also, Dow and Zhou [5] showed that $\beta\omega \setminus \omega$ has a point with local Noetherian type ω. (An easier construction of such a point will be given in the proof of Theorem 5.16, which is a generalization a construction of Isbell [12].)

Observation 1.4. *Every known homogeneous compactum X satisfies the following.*

    (1) $Nt(X) \leq \mathfrak{c}^+$.
    (2) $\pi Nt(X) \leq \omega_1$.
    (3) $\chi Nt(X) = \omega$.
    (4) $\chi_K Nt(X) \leq \mathfrak{c}$.

We justify this observation in Section 2, except that we postpone the case of homogeneous dyadic compacta to Section 3, where we investigate Noetherian cardinal functions on dyadic compacta in general. The results relevant to Observation 1.4 are summarized by the following theorem.

Theorem 1.5. *Suppose X is a dyadic compactum. Then $\pi Nt(X) = \chi_K Nt(X) = \omega$. Moreover, if X is homogeneous, then $Nt(X) = \omega$.*

Also in Section 3, we generalize the above theorem to continuous images of products of compacta with bounded weight; we also prove the following:

Theorem 1.6. *The class of Noetherian types of dyadic compacta includes ω, excludes ω_1, includes all singular cardinals, and includes κ^+ for all cardinals κ with uncountable cofinality.*

Section 4 generalizes our results about dyadic compacta to the proper superclass of k-adic compacta.

Finally, in Section 5, we prove several results about the local Noetherian types of all homogeneous compacta, known and unknown, including the following theorem.

Theorem 1.7 (GCH). *If X is a homogeneous compactum, then $\chi Nt(X) \leq c(X)$.*

2. Observed upper bounds on Noetherian cardinal functions

First, we note some very basic facts about Noetherian cardinal functions.

Definition 2.1. Given a subset E of a product $\prod_{i \in I} X_i$ and $\sigma \in [I]^{<\omega}$, we say that E has *support* σ, or $\mathrm{supp}(E) = \sigma$, if $E = \pi_\sigma^{-1}\pi_\sigma[E]$ and $E \neq \pi_\tau^{-1}\pi_\tau[E]$ for all $\tau \subsetneq \sigma$.

Theorem 2.2. *Given a point p and a compact subset K of a product space $X = \prod_{i \in I} X_i$, we have the following relations.*

$$Nt(X) \leq \sup_{i \in I} Nt(X_i)$$

$$\pi Nt(X) \leq \sup_{i \in I} \pi Nt(X_i)$$

$$\chi Nt(p, X) \leq \sup_{i \in I} \chi Nt(p(i), X_i)$$

$$\chi Nt(K, X) \leq \sup_{\sigma \in [I]^{<\omega}} \chi Nt(\pi_\sigma[K], \pi_\sigma[X])$$

Proof. See Peregudov [23] for a proof of the first relation. That proof can be easily modified to demonstrate the next two relations. Let us prove the last relation. For each $\sigma \in [I]^{<\omega}$, set $\kappa_\sigma = \chi Nt(\pi_\sigma[K], \pi_\sigma[X])$ and let $\mathcal{A}_\sigma$ be a $\kappa_\sigma^{\mathrm{op}}$-like neighborhood base of $\pi_\sigma[K]$. For each $\sigma \in [I]^{<\omega}$, let $\mathcal{B}_\sigma$ denote the set of sets of the form $\pi_\sigma^{-1}U$ where $U \in \mathcal{A}_\sigma$ and $\mathrm{supp}(U) = \sigma$. Note that if $U \in \mathcal{A}_\sigma$ and $\mathrm{supp}(U) \subsetneq \sigma$, then there exists $\tau \subsetneq \sigma$ and $V \in \mathcal{A}_\tau$ such that $\pi_\tau^{-1}V \subseteq \pi_\sigma^{-1}U$. Moreover, for any minimal such τ, we have $\pi_\tau^{-1}V \in \mathcal{B}_\tau$.

Set $\mathcal{B} = \bigcup_{\sigma \in [I]^{<\omega}} \mathcal{B}_\sigma$. By compactness, $\mathcal{B}$ is a neighborhood base of K. Moreover, if $\sigma, \tau \in [I]^{<\omega}$ and $\mathcal{B}_\sigma \ni U \subseteq V \in \mathcal{B}_\tau$, then $\sigma = \mathrm{supp}(U) \supseteq \mathrm{supp}(V) = \tau$; hence, given U, there are at most $(\sup_{\tau \subseteq \sigma} \kappa_\tau)$-many possibilities for V. Thus, $\mathcal{B}$ is $(\sup_{\sigma \in [I]^{<\omega}} \kappa_\sigma)^{\mathrm{op}}$-like as desired. $\square$

Question 2.3. Do there exist spaces X and Y such that $\chi_K Nt(X \times Y)$ exceeds $\chi_K Nt(X) \chi_K Nt(Y)$?

Lemma 2.4. *Every poset P is almost $|P|^{\mathrm{op}}$-like.*

Proof. Let $\kappa = |P|$ and let $\langle p_\alpha \rangle_{\alpha < \kappa}$ enumerate P. Define a partial map $f \colon \kappa \to P$ as follows. Suppose $\alpha < \kappa$ and we have a partial map $f_\alpha \colon \alpha \to P$. If $\mathrm{ran}\, f_\alpha$ is dense in P, then set $f_{\alpha+1} = f_\alpha$. Otherwise, set $\beta = \min\{\delta < \kappa : p_\delta \not\geq q$ for all $q \in \mathrm{ran}\, f_\alpha\}$ and let $f_{\alpha+1}$ be the smallest map extending f_α such that $f_{\alpha+1}(\alpha) = p_\beta$. For limit ordinals $\gamma \leq \kappa$, set $f_\gamma = \bigcup_{\alpha < \gamma} f_\alpha$. Then f_κ is nonincreasing; hence, $\mathrm{ran}\, f_\kappa$ is κ^{op}-like. Moreover, $\mathrm{ran}\, f_\kappa$ is dense in P. $\square$

Theorem 2.5. *For any space X with point p, we have $\chi Nt(p, X) \leq \chi(p, X)$, $\pi Nt(X) \leq \pi(X)$, $Nt(X) \leq w(X)^+$, and $\chi_K Nt(X) \leq w(X)$.*

Proof. The first two relations immediately follow from Lemma 2.4; the third relation is trivial. For the last relation, note that if K is a compact subset of X, then it has neighborhood base of size at most $w(X)$; apply Lemma 2.4. $\square$

Given Theorem 2.2, justifying Observation 1.4 for $Nt(\cdot)$, $\pi Nt(\cdot)$, and $\chi Nt(\cdot)$ amounts to justifying it for first countable homogeneous compacta, dyadic homogeneous compacta, and the two known kinds of exceptional homogeneous compacta. The first countable case is the easiest. By Arhangel′skiĭ's Theorem, first countable compacta have weight at most $\mathfrak{c}$, and therefore have Noetherian type at most $\mathfrak{c}^+$. Moreover, every point in a first countable space clearly has an ω^{op}-like local base. The only nontrivial bound is the one on Noetherian π-type. For that, the following theorem suffices.

Definition 2.6. Give a space X, let $\pi sw(X)$ denote the least κ such that X has a π-base $\mathcal{A}$ such that $\bigcap \mathcal{B} = \emptyset$ for all $\mathcal{B} \in [\mathcal{A}]^{\kappa^+}$.

Theorem 2.7. *If X is a compactum, then $\pi Nt(X) \leq \pi sw(X)^+ \leq t(X)^+ \leq \chi(X)^+$.*

Proof. Only the second relation is nontrivial; it is a theorem of Šapirovskiĭ [26]. $\square$

For dyadic homogeneous compacta, Theorem 1.5 obviously implies Observation 1.4; we will prove this theorem in Section 3. Now consider the two known classes exceptional homogeneous compacta. They are constructed by two techniques, resolutions and amalgams. First we consider the exceptional resolution.

Definition 2.8. Suppose X is a space, $\langle Y_p \rangle_{p \in X}$ is a sequence of nonempty spaces, and $\langle f_p \rangle_{p \in X} \in \prod_{p \in X} C(X \setminus \{p\}, Y_p)$. Then the *resolution* Z of X at each point p into Y_p by f_p is defined by setting $Z = \bigcup_{p \in X}(\{p\} \times Y_p)$ and declaring Z to have weakest topology such that, for every $p \in X$, open neighborhood U of p in X, and open $V \subseteq Y_p$, the set $U \otimes V$ is open in Z where

$$U \otimes V = (\{p\} \times V) \cup \bigcup_{q \in U \cap f_p^{-1} V} (\{q\} \times Y_q).$$

The resolution of concern to us in constructed by van Mill [30]. It is a compactum with weight $\mathfrak{c}$, π-weight ω, and character ω_1. Moreover, assuming $MA + \neg CH$ (or just $\mathfrak{p} > \omega_1$), this space is homogeneous. (It is not homogeneous if $2^\omega < 2^{\omega_1}$.) For a proof that this space is exceptional (assuming $MA + \neg CH$), see [20]. Clearly, this space has sufficiently small Noetherian type and π-type. We just need to show that it has local Noetherian type ω. Van Mill's space is a resolution of 2^ω at each point into $\mathbb{T}^{\omega_1}$ where $\mathbb{T}$ is the circle group $\mathbb{R}/\mathbb{Z}$.

Notice that $\mathbb{T}$ is metrizable. The following lemma proves that every metric compactum has Noetherian type ω, along with some results that will be useful in Section 3.

Lemma 2.9. *Let X be a metric compactum with base $\mathcal{A}$. Then there exists $\mathcal{B} \subseteq \mathcal{A}$ satisfying the following.*

(1) *$\mathcal{B}$ is a base of X.*
(2) *$\mathcal{B}$ is ω^{op}-like.*
(3) *If $U, V \in \mathcal{B}$ and $U \subsetneq V$, then $\overline{U} \subseteq V$.*
(4) *For all $\Gamma \in [\mathcal{B}]^{<\omega}$, there are only finitely many $U \in \mathcal{B}$ such that Γ contains $\{V \in \mathcal{B} : U \subsetneq V\}$.*

Proof. Construct a sequence $\langle \mathcal{B}_n \rangle_{n < \omega}$ of finite subsets of $\mathcal{A}$ as follows. For each $n < \omega$, let E_n be the union of the set of all singletons in $\bigcup_{m < n} \mathcal{B}_m$. Let $\mathcal{C}_n$ be the set of all $U \in \mathcal{A}$ for which $U \cap E_n = \emptyset$ and

$$2^{-n} \geq \operatorname{diam} U < \min\left\{ \operatorname{diam} V : V \in \bigcup_{m < n} \mathcal{B}_m \text{ and } 0 < \operatorname{diam} V \right\}$$

and $\overline{U} \subseteq V$ for all $V \in \bigcup_{m < n} \mathcal{B}_m$ strictly containing U. Then $\bigcup \mathcal{C}_n = X \setminus E_n$. Let $\mathcal{B}_n$ be a minimal finite subcover of $\mathcal{C}_n$. Set $\mathcal{B} = \bigcup_{n < \omega} \mathcal{B}_n$. To prove (3), suppose $U \in \mathcal{B}_n$ and $V \in \mathcal{B}_m$ and $U \subsetneq V$. Then $m \neq n$ by minimality of $\mathcal{B}_n$. Also, $0 < \operatorname{diam} V$ because $\emptyset \neq U \subsetneq V$. Hence, if $m > n$, then $\operatorname{diam} V < \operatorname{diam} U$, in contradiction with $U \subsetneq V$. Hence, $m < n$; hence, $\overline{U} \subseteq V$.

For (1), let $p \in X$ and $n < \omega$, and let V be the open ball with radius 2^{-n} and center p. Then we just need to show that there exists $U \in \mathcal{B}$ such that $p \in U \subseteq V$.

Hence, we may assume $\{p\} \notin \mathcal{B}$. Hence, $p \notin E_{n+1}$; hence, there exists $U \in \mathcal{B}_{n+1}$ such that $p \in U$. Since $\operatorname{diam} U \leq 2^{-n-1}$, we have $U \subseteq V$.

For (2), let $n < \omega$ and $U \in \mathcal{B}_n$. If U is a singleton, then every superset of U in $\mathcal{B}$ is in $\bigcup_{m \leq n} \mathcal{B}_m$. If U is not a singleton, then U has diamater at least 2^{-m} for some $m < \omega$; whence, every superset of U in $\mathcal{B}$ is in $\bigcup_{l \leq m} \mathcal{B}_l$.

For (4), suppose $\Gamma \in [\mathcal{B}]^{<\omega}$ and there exist infinitely many $U \in \mathcal{B}$ such that $\{V \in \mathcal{B} : U \subsetneq V\} \subseteq \Gamma$. We may assume Γ contains no singletons. Choose an increasing sequence $\langle k_n \rangle_{n<\omega}$ in ω such that, for all $n < \omega$, there exists $U_n \in \mathcal{B}_{k_n}$ such that $\{V \in \mathcal{B} : U_n \subsetneq V\} \subseteq \Gamma$. For each $n < \omega$, choose $p_n \in U_n$. Since $\{U_n : n < \omega\}$ is infinite, we may choose $\langle p_n \rangle_{n<\omega}$ such that $\{p_n : n < \omega\}$ is infinite. Let p be an accumulation point of $\{p_n : n < \omega\}$. Choose $m < \omega$ such that $2^{-m} < \operatorname{diam} V$ for all $V \in \Gamma$. Since p is not an isolated point, there exists $W \in \mathcal{B}_m$ such that $p \in W$. Then $W \notin \Gamma$; hence, W does not strictly contain U_n for any $n < \omega$. Choose $q \in W \setminus \{p\}$ such that W contains $\{x : d(p,x) \leq d(p,q)\}$; set $r = d(p,q)$. Let B be the open ball of radius $r/2$ centered about p. Then there exists $n < \omega$ such that $2^{-k_n} < r/2$ and $p_n \in B$. Hence, $\operatorname{diam} U_n < r/2$ and $U_n \cap B \neq \emptyset$; hence, $U_n \subseteq W$ and $q \notin U_n$; hence, $U_n \subsetneq W$, which is absurd. Therefore, for each $\Gamma \in [\mathcal{B}]^{<\omega}$, there are only finitely many $U \in \mathcal{B}$ such that $\{V \in \mathcal{B} : U \subsetneq V\} \subseteq \Gamma$. □

We have $Nt(2^\omega) = Nt(\mathbb{T}^{\omega_1}) = \omega$ by Lemma 2.9 and Theorem 2.2. Therefore, the following theorem implies that van Mill's space has local Noetherian type ω.

Lemma 2.10 ([30])**.** *Suppose X, $\langle Y_p \rangle_{p \in X}$, $\langle f_p \rangle_{p \in X}$, and Z are as in Definition 2.8. Suppose $\mathcal{U}$ is a local base at a point p in X and $\mathcal{V}$ is a local base at a point y in Y_p. Then $\{U \otimes V : \langle U, V \rangle \in \mathcal{U} \times (\mathcal{V} \cup \{Y_p\})\}$ is a local base at $\langle p, y \rangle$ in Z.*

Theorem 2.11. *Suppose X, $\langle Y_p \rangle_{p \in X}$, $\langle f_p \rangle_{p \in X}$, and Z are as in Definition 2.8. Then $\chi Nt(\langle p, y \rangle, Z) \leq Nt(X) \chi Nt(y, Y_p)$ for all $\langle p, y \rangle \in Z$.*

Proof. Set $\kappa = Nt(X) \chi Nt(y, Y_p)$. Let $\mathcal{A}$ be a κ^{op}-like base of X and let $\mathcal{B}$ be a κ^{op}-like local base at y in Y_p; we may assume $Y_p \in \mathcal{B}$. Set $\mathcal{C} = \{U \in \mathcal{A} : p \in U\}$. Set $\mathcal{D} = \{U \otimes V : \langle U, V \rangle \in \mathcal{C} \times \mathcal{B}\}$, which is a local base at $\langle p, y \rangle$ in Z by Lemma 2.10. If there exists $U \otimes V \in \mathcal{D}$ such that $U \cap f_p^{-1} V = \emptyset$, then $U \otimes V$ is homeomorphic to V; whence, $\chi Nt(\langle p, y \rangle, Z) = \chi Nt(y, Y_p) \leq \kappa$. Hence, we may assume $U \cap f_p^{-1} V \neq \emptyset$ for all $U \otimes V \in \mathcal{D}$.

It suffices to show that $\mathcal{D}$ is κ^{op}-like. Suppose $U_i \otimes V_i \in \mathcal{D}$ for all $i < 2$ and $U_0 \otimes V_0 \subseteq U_1 \otimes V_1$. Then $V_0 \subseteq V_1$ and $\emptyset \neq U_0 \cap f_p^{-1} V_0 \subseteq U_1 \cap f_p^{-1} V_1$. Since $\mathcal{B}$ is κ^{op}-like, there are fewer than κ-many possibilities for V_1 given V_0. Since $\mathcal{A}$ is a κ^{op}-like base, there are fewer than κ-many possibilities for U_1 given U_0 and V_0. Hence, there are fewer than κ-many possibilities for $U_1 \otimes V_1$ given $U_0 \otimes V_0$. □

Definition 2.12. Let $\mathfrak{p}$ denote the least κ for which some $\mathcal{A} \in [[\omega]^\omega]^\kappa$ has the strong finite intersection property but does not have a nontrivial pseudointersection. By a theorem of Bell [4], $\mathfrak{p}$ is also the least κ for which there exist a σ-centered poset $\mathbb{P}$ and a family $\mathcal{D}$ of κ-many dense subsets of $\mathbb{P}$ such that $\mathbb{P}$ does not have a $\mathcal{D}$-generic filter.

Definition 2.13. Given a space X, let $\operatorname{Aut}(X)$ denote the set of its autohomeomorphisms.

Van Mill's construction has been generalized by Hart and Ridderbos [10]. They show that one can produce an exceptional homogeneous compactum with weight $\mathfrak{c}$

and π-weight ω by carefully resolving each point of 2^ω into a fixed space Y satisfying the following conditions.

(1) Y is a homogeneous compactum.
(2) $\omega_1 \leq \chi(Y) \leq w(Y) < \mathfrak{p}$.
(3) $\exists d \in Y \quad \exists \eta \in \mathrm{Aut}(Y) \quad \overline{\{\eta^n(d) : n < \omega\}} = Y$.
(4) If $\gamma\omega$ is a compactification of ω and $\gamma\omega \setminus \omega \cong Y$, then Y is a retract of $\gamma\omega$.

By Theorem 2.11, to show that such resolutions have local Noetherian type ω, it suffices to show that every such Y has local Noetherian type ω. Theorem 2.16 will accomplish this.

Theorem 2.14. *Suppose X is a compactum and $\pi\chi(p, X) = \chi(q, X)$ for all $p, q \in X$. Then $\chi Nt(p, X) = \omega$ for some $p \in X$. In particular, if X is a homogeneous compactum and $\pi\chi(X) = \chi(X)$, then $\chi Nt(X) = \omega$.*

The proof of Theorem 2.14 will be delayed until Section 5.

The following lemma is essentially a generalization of a similar result of Juhász [15].

Lemma 2.15. *Suppose X is a compactum and $\omega = d(X) \leq w(X) < \mathfrak{p}$. Then there exists $p \in X$ such that $\chi(p, X) \leq \pi(X)$.*

Proof. Let $\mathcal{A}$ be a base of X of size at most $w(X)$. Let $\mathcal{B}$ be a π-base of X of size at most $\pi(X)$. For each $\langle U, V \rangle \in \mathcal{B}^2$ satisfying $\overline{U} \subseteq V$, choose a closed G_δ-set $\Phi(U, V)$ such that $\overline{U} \subseteq \Phi(U, V) \subseteq V$. Then $\mathrm{ran}\,\Phi$, ordered by $\subseteq$, is σ-centered because $d(X) = \omega$. Since $|\mathcal{A}| < \mathfrak{p}$, there is a filter $\mathcal{G}$ of $\mathrm{ran}\,\Phi$ such that for all disjoint $U, V \in \mathcal{A}$ some $K \in \mathcal{G}$ satisfies $U \cap K = \emptyset$ or $V \cap K = \emptyset$. Hence, there exists a unique $p \in \bigcap \mathcal{G}$. Hence, p has pseudocharacter, and therefore character, at most $|\mathcal{G}|$, which is at most $\pi(X)$. $\qquad\square$

Theorem 2.16. *If X is a homogeneous compactum and $\omega = d(X) \leq w(X) < \mathfrak{p}$, then $\chi Nt(X) = \omega$.*

Proof. By Lemma 2.15, $\chi(X) \leq \pi(X) = \pi\chi(X)d(X) = \pi\chi(X)$. Hence, by Theorem 2.14, $\chi Nt(X) = \omega$. $\qquad\square$

Amalgams are defined in [20] as follows.

Definition 2.17. Suppose X is a T_0 space, $\mathscr{S}$ is a subbase of X such that $\emptyset \notin \mathscr{S}$, and $\langle Y_S \rangle_{S \in \mathscr{S}}$ is a sequence of nonempty spaces. The *amalgam* Y of $\langle Y_S : S \in \mathscr{S} \rangle$ is defined by setting $Y = \bigcup_{p \in X} \prod_{p \in S \in \mathscr{S}} Y_S$ and declaring Y to have the weakest topology such that, for each $S \in \mathscr{S}$ and open $U \subseteq Y_S$, the set $\pi_S^{-1}U$ is open in Y where $\pi_S^{-1}U = \{p \in Y : S \in \mathrm{dom}\,p \text{ and } p(S) \in U\}$. Define $\pi : Y \to X$ by $\{\pi(p)\} = \bigcap \mathrm{dom}\,p$ for all $p \in Y$. It is easily verified that π is continuous.

Theorem 2.18. *Suppose X, $\mathscr{S}$, $\langle Y_S \rangle_{S \in \mathscr{S}}$, and Y be as in Definition 2.17. Then we have the following relations for all $p \in Y$.*

$$Nt(Y) \leq Nt(X) \sup_{S \in \mathscr{S}} Nt(Y_S)$$

$$\pi Nt(Y) \leq \pi Nt(X) \sup_{S \in \mathscr{S}} \pi Nt(Y_S)$$

$$\chi Nt(p, Y) \leq \chi Nt(\pi(p), X) \sup_{S \in \mathrm{dom}\,p} \chi Nt(p(S), Y_S)$$

Proof. We will only prove the first relation; the proofs of the others are almost identical. Set $\kappa = Nt(X) \sup_{S \in \mathscr{S}} Nt(Y_S)$. Let $\mathcal{A}$ be a κ^{op}-like base of X. For each $S \in \mathscr{S}$, let $\mathcal{B}_S$ be a κ^{op}-like base of Y_S. Set

$$\mathcal{C} = \left\{ \pi^{-1}U \cap \bigcap_{S \in \mathrm{dom}\,\tau} \pi_S^{-1}\tau(S) : \tau \in \bigcup_{\mathcal{F} \in [\mathscr{S}]^{<\omega}} \prod_{S \in \mathcal{F}} \mathcal{B}_S \setminus \{Y_S\} \text{ and } \mathcal{A} \ni U \subseteq \bigcap \mathrm{dom}\,\tau \right\}.$$

Then $\mathcal{C}$ is clearly a base of Y. Let us show that $\mathcal{C}$ is κ^{op}-like. Suppose $\pi^{-1}U_i \cap \bigcap_{S \in \mathrm{dom}\,\tau_i} \pi_S^{-1}\tau_i(S) \in \mathcal{C}$ for all $i < 2$ and

$$\pi^{-1}U_0 \cap \bigcap_{S \in \mathrm{dom}\,\tau_0} \pi_S^{-1}\tau_0(S) \subseteq \pi^{-1}U_1 \cap \bigcap_{S \in \mathrm{dom}\,\tau_1} \pi_S^{-1}\tau_1(S).$$

Then $U_0 \subseteq U_1$ and $\mathrm{dom}\,\tau_0 \supseteq \mathrm{dom}\,\tau_1$ and $\tau_0(S) \subseteq \tau_1(S)$ for all $S \in \mathrm{dom}\,\tau_1$. Hence, there are fewer than κ-many possibilities for U_1 and τ_1 given U_0 and τ_0. $\qquad \square$

An exceptional homogeneous compactum Y is constructed in [20] with $X = \mathbb{T}$ and $w(Y_S) = \pi(Y_S) = \mathfrak{c}$ and $\chi(Y_S) = \omega$ for all $S \in \mathscr{S}$. Hence, $Nt(Y_S) \leq \mathfrak{c}^+$ and $\chi Nt(Y_S) = \omega$ for each $S \in \mathscr{S}$. Moreover, each Y_S is 2^γ ordered lexicographically where γ is a fixed indecomposable ordinal in $\omega_1 \setminus (\omega + 1)$. Since $\mathrm{cf}\,\gamma = \omega$, it is easy to construct an ω^{op}-like π-base of this space. Hence, by Theorem 2.18, $Nt(Y) \leq \mathfrak{c}^+$ and $\pi Nt(Y) = \chi Nt(Y) = \omega$. Thus, Observation 1.4 is justified for $Nt(\cdot)$, $\pi Nt(\cdot)$, and $\chi Nt(\cdot)$.

It remains to justify Observation 1.4 for $\chi_K Nt(\cdot)$. We first note that all known homogeneous compacta are continuous images of products of compacta each of weight at most $\mathfrak{c}$. (Moreover, it it shown in [20] that any Z as in Definition 2.17 is a continuous image of $X \times \prod_{S \in \mathscr{S}} Y_S$.) Therefore, the following theorem will suffice.

Theorem 2.19. *Suppose Y is a continuous image of a product $X = \prod_{i \in I} X_i$ of compacta. Then $\chi_K Nt(Y) \leq \sup_{i \in I} w(X_i)$*

Before proving the above theorem, we first prove two lemmas.

Definition 2.20. Given subsets P and Q of a common poset, define P and Q to be *mutually dense* if for all $p_0 \in P$ and $q_0 \in Q$ there exist $p_1 \in P$ and $q_1 \in Q$ such that $p_0 \geq q_1$ and $q_0 \geq p_1$.

Lemma 2.21. *Let κ be a cardinal and let P and Q be mutually dense subsets of a common poset. Then P is almost κ^{op}-like if and only if Q is.*

Proof. Suppose D is a κ^{op}-like dense subset of P. Then it suffices to construct a κ^{op}-like dense subset of Q. Define a partial map f from $|D|^+$ to Q as follows. Set $f_0 = \emptyset$. Suppose $\alpha < |D|^+$ and we have constructed a partial map f_α from α to Q. Set $E = \{d \in D : d \not\geq q \text{ for all } q \in \mathrm{ran}\,f_\alpha\}$. If $E = \emptyset$, then set $f_{\alpha+1} = f_\alpha$. Otherwise, choose $q \in Q$ such that $q \leq e$ for some $e \in E$, and let $f_{\alpha+1}$ be the smallest function extending f_α such that $f_{\alpha+1}(\alpha) = q$. For limit ordinals $\gamma \leq |D|^+$, set $f_\gamma = \bigcup_{\alpha < \gamma} f_\alpha$. Set $f = f_{|D|^+}$.

Let us show that $\mathrm{ran}\,f$ is κ^{op}-like. Suppose otherwise. Then there exists $q \in \mathrm{ran}\,f$ and an increasing sequence $\langle \xi_\alpha \rangle_{\alpha < \kappa}$ in $\mathrm{dom}\,f$ such that $q \leq f(\xi_\alpha)$ for all $\alpha < \kappa$. By the way we constructed f, there exists $\langle d_\alpha \rangle_{\alpha < \kappa} \in D^\kappa$ such that $f(\xi_\beta) \leq d_\beta \neq d_\alpha$ for all $\alpha < \beta < \kappa$. Choose $p \in P$ such that $p \leq q$. Then choose $d \in D$ such that $d \leq p$. Then $d \leq d_\beta \neq d_\alpha$ for all $\alpha < \beta < \kappa$, which contradicts that D is κ^{op}-like. Therefore, $\mathrm{ran}\,f$ is κ^{op}-like.

Finally, let us show that $\operatorname{ran} f$ is a dense subset of Q. Suppose $q \in Q$. Choose $p \in P$ such that $p \leq q$. Then choose $d \in D$ such that $d \leq p$. By the way we constructed f, there exists $r \in \operatorname{ran} f$ such that $r \leq d$; hence, $r \leq q$. $\qquad \square$

Lemma 2.22. *Suppose* $f \colon X \to Y$ *is a continuous surjection between compacta and C is closed in Y. Then* $\chi Nt(f^{-1}C, X) = \chi Nt(C, Y)$.

Proof. Let $\mathcal{A}$ be a neighborhood base of C. By Lemma 2.21, it suffices to show that $\{f^{-1}V : V \in \mathcal{A}\}$ is a neighborhood base of $f^{-1}C$. Suppose U is a neighborhood of $f^{-1}C$. By normality of Y, we have $f^{-1}C = \bigcap_{V \in \mathcal{A}} f^{-1}\overline{V}$. By compactness of X, we have $f^{-1}\overline{V} \subseteq U$ for some $V \in \mathcal{A}$. Thus, $\{f^{-1}V : V \in \mathcal{A}\}$ is a neighborhood base of $f^{-1}C$ as desired. $\qquad \square$

Proof of Theorem 2.19. By Lemma 2.22, we may assume $Y = X$. By Theorem 2.2, we may assume I is finite. Apply Theorem 2.5. $\qquad \square$

How sharp are the bounds of Observation 1.4? (3) is trivially sharp as every space has local Noetherian type at least ω. We will show that there is a homogeneous compactum with Noethian type $\mathfrak{c}^+$, namely, the double arrow space. Moreover, we will show that Suslin lines have uncountable Noetherian π-type. It is known to be consistent that there are homogeneous compact Suslin lines, but it is also known to be consistent that there are no Suslin lines. It is not clear whether it is consistent that all homogeneous compacta have Noetherian π-type ω, even if we restrict to the first countable case. Also, it is not clear in any model of ZFC whether all first countable homogeneous compacta have compact Noetherian type ω.

Question 2.23. Is there a first countable compactum with uncountable compact Noetherian type?

The following proposition is essentially due to Peregudov [23].

Proposition 2.24. *If X is a space and $\pi(X) < \operatorname{cf} \kappa \leq \kappa \leq w(X)$, then $Nt(X) > \kappa$.*

Proof. Suppose $\mathcal{A}$ is a base of X and $\mathcal{B}$ is π-base of X of size $\pi(X)$. Then $|\mathcal{A}| \geq \kappa$; hence, there exist $\mathcal{U} \in [\mathcal{A}]^\kappa$ and $V \in \mathcal{B}$ such that $V \subseteq \bigcap \mathcal{U}$. Hence, there exists $W \in \mathcal{A}$ such that $W \subseteq V \subseteq \bigcap \mathcal{U}$; hence, $\mathcal{A}$ is not κ^{op}-like. $\qquad \square$

Example 2.25. The double arrow space, defined as $((0, 1] \times \{0\}) \cup ([0, 1) \times \{1\})$ ordered lexicographically, has π-weight ω and weight $\mathfrak{c}$, and is known to be compact and homogeneous. By Proposition 2.24, it has Noetherian type $\mathfrak{c}^+$.

Theorem 2.26. *Suppose X is a Suslin line. Then* $\pi Nt(X) \geq \omega_1$.

Proof. Let $\mathcal{A}$ be a π-base of X consisting only of open intervals. By Lemma 2.21, it suffices to show that $\mathcal{A}$ is not ω^{op}-like. Construct a sequence $\langle \mathcal{B}_n \rangle_{n < \omega}$ of maximal pairwise disjoint subsets of $\mathcal{A}$ as follows. Choose $\mathcal{B}_0$ arbitrarily. Given $n < \omega$ and $\mathcal{B}_n$, choose $\mathcal{B}_{n+1}$ such that it refines $\mathcal{B}_n$ and $\mathcal{B}_n \cap \mathcal{B}_{n+1} \subseteq [X]^1$.

Let E denote the set of all endpoints of intervals in $\bigcup_{n < \omega} \mathcal{B}_n$. Since X is Suslin, there exists $U \in \mathcal{A} \setminus [X]^1$ such that $U \cap E = \emptyset$. For each $n < \omega$, the set $\bigcup \mathcal{B}_n$ is dense in X by maximality; whence, there exists $V_n \in \mathcal{B}_n$ such that $U \cap V_n \neq \emptyset$. Since $U \cap E = \emptyset$, we have $U \subseteq \bigcap_{n < \omega} V_n$. Thus, $\mathcal{A}$ is not ω^{op}-like. $\qquad \square$

MA + ¬CH implies there are no Souslin lines. It is not clear whether it further implies every homogeneous compactum has Noetherian π-type ω. However, the

next theorem gives us a partial result. First, we need a lemma very similar to the result that MA + ¬CH implies all Aronszajn trees are special.

Definition 2.27. Given a subset E of a poset Q, let $\uparrow_Q E$ denote the set of $q \in Q$ for which q has a lower bound in E.

Lemma 2.28. *Assume MA. Suppose Q is an ω_1^{op}-like poset of size less than $\mathfrak{c}$. Then Q is almost ω^{op}-like or Q has an uncountable centered subset.*

Proof. Set $\mathbb{P} = [Q]^{<\omega}$ and order $\mathbb{P}$ such that $\sigma \leq \tau$ if and only if $\sigma \cap \uparrow_Q \tau = \tau$. A sufficiently generic filter G of $\mathbb{P}$ will be such that $\bigcup G$ is a dense ω^{op}-like subset of Q. Hence, if $\mathbb{P}$ is ccc, then Q is almost ω^{op}-like. Hence, we may assume $\mathbb{P}$ has an antichain A of size ω_1. We may assume A is a Δ-system with root ρ. Since Q is ω_1^{op}-like, we may assume $\sigma \cap \uparrow_Q \rho = \rho$ for all $\sigma \in A$. Choose a bijection $\langle a_\alpha \rangle_{\alpha < \omega_1}$ from ω_1 to A. We may assume there exists an $n < \omega$ such that $|a_\alpha \setminus \rho| = n$ for all $\alpha < \omega_1$. For each $\alpha < \omega_1$, choose a bijection $\langle a_{\alpha,i} \rangle_{i<n}$ from n to $a_\alpha \setminus \rho$. For each $x \in Q$ and $i < n$, set $E_{x,i} = \{\alpha < \omega_1 : x \leq_Q a_{\alpha,i} \text{ or } a_{\alpha,i} \leq_Q x\}$. For each $\alpha < \omega_1$, since A is an antichain, we have $\bigcup_{i<n} \bigcup_{j<n} E_{a_{\alpha,i},j} = \omega_1$. Choose a uniform ultrafilter $\mathcal{U}$ on ω_1. Then we may choose $B \in [(\bigcup A) \setminus \rho]^{\omega_1}$ and $i < n$ such that $E_{x,i} \in \mathcal{U}$ for all $x \in B$.

It suffices to show that B is centered. Let $\sigma \in [B]^{<\omega}$. Set $E = \bigcap_{x \in \sigma} E_{x,i}$. Then $E \in \mathcal{U}$; hence, $|E| = \omega_1$; hence, we may choose $\alpha \in E \setminus \{\beta < \omega_1 : a_{\beta,i} \in \uparrow_Q \sigma\}$. Then $a_{\alpha,i} <_Q x$ for all $x \in \sigma$. Thus, B is centered. □

Lemma 2.29. *Suppose $f \colon X \to Y$ is an irreducible continuous surjection between spaces and X is regular. Then $\pi Nt(X) = \pi Nt(Y)$.*

Proof. Let $\mathcal{A}$ be a $\pi Nt(X)^{\mathrm{op}}$-like π-base of X and let $\mathcal{B}$ be a $\pi Nt(Y)^{\mathrm{op}}$-like π-base of Y. By Lemma 2.21, we may assume $\mathcal{A}$ consists only of regular open sets. Set $\mathcal{C} = \{f^{-1}U : U \in \mathcal{B}\}$. Then $\mathcal{C}$ is $\pi Nt(Y)^{\mathrm{op}}$-like. Suppose U is a nonempty open subset of X. Then we may choose $V \in \mathcal{B}$ such that $V \cap f[X \setminus U] = \emptyset$. Then $f^{-1}V \subseteq U$. Thus, $\mathcal{C}$ is a π-base of X; hence, $\pi Nt(X) \leq \pi Nt(Y)$.

Set $\mathcal{D} = \{Y \setminus f[X \setminus U] : U \in \mathcal{A}\}$. Suppose V is a nonempty open subset of Y. Then we may choose $U \in \mathcal{A}$ such that $U \subseteq f^{-1}V$. Then $Y \setminus f[X \setminus U] \subseteq V$. Thus, $\mathcal{D}$ is a π-base of Y. Now suppose $U_0, U_1 \in \mathcal{A}$ and $U_0 \nsubseteq U_1$. Then $U_0 \nsubseteq \overline{U}_1$ by regularity. By irreducibility, we may choose $p \in Y \setminus f[X \setminus (U_0 \setminus \overline{U}_1)]$. Then $p \in f[X \setminus U_1]$ and $p \notin f[X \setminus U_0]$. Hence, $Y \setminus f[X \setminus U_0] \nsubseteq Y \setminus f[X \setminus U_1]$. Thus, $\mathcal{D}$ is $\pi Nt(X)^{\mathrm{op}}$-like; hence, $\pi Nt(Y) \leq \pi Nt(X)$. □

Theorem 2.30. *Assume MA. Let X be a compactum such that $t(X) = \omega$ and $\pi(X) < \mathfrak{c}$. Then $\pi Nt(X) = \omega$.*

Proof. We may assume X is a closed subspace of $[0,1]^\kappa$ for some cardinal κ. By a result of Šapirovskiĭ [26], since $t(X) = \omega$, there is an irreducible continuous map f from X onto a subspace of $\bigcup_{I \in [\kappa]^\omega} [0,1]^I \times \{0\}^{\kappa \setminus I}$. Because of Lemma 2.29, we may replace our hypothesis of $t(X) = \omega$ with $X \subseteq \bigcup_{I \in [\kappa]^\omega} [0,1]^I \times \{0\}^{\kappa \setminus I}$. Set $\mathcal{F} = \mathrm{Fn}(\kappa, (\mathbb{Q} \cap (0,1])^2)$ and

$$\mathcal{A} = \left\{ X \cap \bigcap_{\alpha \in \mathrm{dom}\, \sigma} \pi_\alpha^{-1}(\sigma(\alpha)(0), \sigma(\alpha)(1)) : \sigma \in \mathcal{F} \right\} \setminus \{\emptyset\},$$

which is a π-base of X. Then $\mathcal{A}$ witnesses that $\pi sw(X) = \omega$. Hence, by Theorem 2.7 and Lemma 2.21, $\mathcal{A}$ contains an ω_1^{op}-like dense subset $\mathcal{B}$, and it suffices to show that

$\mathcal{B}$ is almost ω^{op}-like. Seeking a contradiction, suppose $\mathcal{B}$ is not almost ω^{op}-like. By Lemma 2.28, $\mathcal{B}$ contains an uncountable centered subset $\mathcal{C}$. Let the map

$$\left\langle X \cap \bigcap_{\alpha \in \mathrm{dom}\, \sigma_\beta} \pi_\alpha^{-1}(\sigma_\beta(\alpha)(0), \sigma_\beta(\alpha)(1)) \right\rangle_{\beta < \omega_1}$$

be an injection from ω_1 to $\mathcal{C}$. Then $|\bigcup_{\beta < \omega_1} \mathrm{dom}\, \sigma_\beta| = \omega_1$. By compactness, the set

$$X \cap \bigcap_{\beta < \omega_1} \bigcap_{\alpha \in \mathrm{dom}\, \sigma_\beta} \pi_\alpha^{-1}[\sigma_\beta(\alpha)(0), \sigma_\beta(\alpha)(1)]$$

is nonempty, in contradiction with $X \subseteq \bigcup_{I \in [\kappa]^\omega} [0,1]^I \times \{0\}^{\kappa \setminus I}$. $\square$

Concerning compact Noetherian type, we note that if there is a homogeneous compactum X for which $\chi_K Nt(X) \geq \omega_1$, then X is not an ordered space.

Definition 2.31. A point p in a space X is P_κ-*point* if, for every set $\mathcal{A}$ of fewer than κ-many neighborhoods of p, the set $\bigcap \mathcal{A}$ has p in its interior. A P-point is a P_{ω_1}-point.

Theorem 2.32. *If X is a homogeneous ordered compactum, then $\chi_K Nt(X) = \omega$.*

Proof. We may assume X is infinite; hence, X has a point that is not a P-point. By homogeneity, $\min X$ is not a P-point; hence, $\min X$ has countable character. By homogeneity, X is first countable. Let C be closed in X. Then $X \setminus C$ is a disjoint union of open intervals $\bigcup_{i \in I}(a_i, b_i)$ such that $(a_i, b_i) = \bigcup_{n < \omega}[a_{i,n}, b_{i,n}]$ and $\langle a_{i,n} \rangle_{n < \omega}$ is nonincreasing and $\langle b_{i,n} \rangle_{n < \omega}$ is nondecreasing for all $i \in I$. Hence, $\{X \setminus \bigcup_{i \in \mathrm{dom}\, \sigma}[a_{i,\sigma(i)}, b_{i,\sigma(i)}] : \sigma \in \mathrm{Fn}(I, \omega)\}$ is an ω^{op}-like neighborhood base of C. $\square$

It is worth noting that while products do not decrease cellularity, they can decrease $Nt(\cdot)$, $\pi Nt(\cdot)$, and $\chi Nt(\cdot)$, as shown by the following theorem of Malykhin [18].

Theorem 2.33. *Let $p \in X = \prod_{i \in I} X_i$ where X_i is a nonsingleton T_1 space for all $i \in I$. If $\sup_{i \in I} w(X_i) \leq |I|$, then $Nt(X) = \omega$. If $\sup_{i \in I} \pi(X_i) \leq |I|$, then $\pi Nt(X) = \omega$. If $\sup_{i \in I} \chi(p(i), X_i) \leq |I|$, then $\chi Nt(p, X) = \omega$.*

Proof. See [18] for a proof of the first implication. That proof can be easily modified to demonstrate the other implications. $\square$

In constrast, $\chi_K Nt(\cdot)$ is not decreased by products when the factors are compacta. Just as is true of cellularity, the compact Noetherian type of a product of compacta is the supremum of the compact Noetherian types of its finite subproducts.

Theorem 2.34. *If $X = \prod_{i \in I} X_i$ is a product of compacta, then $\chi_K Nt(X) = \sup_{\sigma \in [I]^{<\omega}} \chi_K Nt(\prod_{i \in \sigma} X_i)$.*

Proof. To prove "$\leq$", apply Theorem 2.2. To prove "$\geq$", apply Lemma 2.22. $\square$

Though cellularity and compact Noetherian type behave similarly for compacta, they do not coincide, even assuming homogeneity. Given any indecomposable ordinal γ strictly between ω and ω_1, the lexicographic ordering of 2^γ is homogeneous and compact and has cellularity $\mathfrak{c}$ by a result of Maurice [19]. However, by Theorem 2.32, this space has compact Noetherian type ω.

3. Dyadic compacta

In this section, we prove a strengthened version of Theorem 1.5 and generalize it to continuous images of products of compacta with bounded weight. We also investigate the spectrum of Noetherian types of dyadic compacta. Our approach is to start with results about subsets of free boolean algebras and then use Stone duality to apply them to families of open subsets of dyadic compacta.

By Lemma 2.4, every countable subset of a free boolean algebra is almost ω^{op}-like. We wish to prove this for all subsets of free boolean algebras. We achieve this by approximating free boolean algebras by smaller free subalgebras using elementary substructures. More specifically, we use elementary submodels of H_θ where θ is a regular cardinal and H_θ is the $\{\in\}$-structure of the family of sets that hereditarily have size less than θ. Whenever we use H_θ in an argument, we implicitly assume that θ is sufficiently large to make the argument valid. As is typical with elementary submodels of H_θ, we need reflection properties. For our purposes, the crucial reflection property of free boolean algebras is given by the following lemma.

Lemma 3.1. *Let B be a free boolean algebra and let $\{B, \wedge, \vee\} \subseteq M \prec H_\theta$. Then, for all $q \in B$, there exists $r \in B \cap M$ such that, for all $p \in B \cap M$, we have $p \geq q$ if and only if $p \geq r$. In particular, $r \geq q$.*

Proof. Let $q \in B$. We may assume $q \neq 0$. By elementarity, there exists a map $g \in M$ enumerating a set of mutually independent generators of B. Set $G = \bigcup\{\{g(i), g(i)'\} : i \in \mathrm{dom}\, g\}$. Then there exists $\eta \in [[G]^{<\omega}]^{<\omega}$ such that $q = \bigvee_{\tau \in \eta} \bigwedge \tau$ and $\bigwedge \tau \neq 0$ for all $\tau \in \eta$. Set $r = \bigvee_{\tau \in \eta} \bigwedge (\tau \cap M)$. Let $p \in B \cap M$; we may assume $p \neq 1$. Then there exists $\zeta \in [[G \cap M]^{<\omega}]^{<\omega}$ such that $p = \bigwedge_{\sigma \in \zeta} \bigvee \sigma$ and $\bigvee \sigma \neq 1$ for all $\sigma \in \zeta$. Hence, $p \geq q$ iff, for all $\sigma \in \zeta$ and $\tau \in \eta$, we have $\bigvee \sigma \geq \bigwedge \tau$, which is equivalent to $\sigma \cap \tau \neq \emptyset$, which is equivalent to $\sigma \cap \tau \cap M \neq \emptyset$. Thus, $p \geq q$ if and only if $p \geq r$. $\square$

The above lemma is not new. Fuchino proved that the conclusion of the above lemma is equivalent to the Freese-Nation property, a property free boolean algebras are known to have. (See section 2.2 and Theorem A.2.1 of [11] for details.)

Theorem 3.2. *Every subset of every free boolean algebra is almost ω^{op}-like.*

Proof. Let B be a free boolean algebra; set $\kappa = |B|$. Given $A \subseteq B$, let $\uparrow A$ denote the smallest semifilter of B containing A; if $A = \{a\}$ for some a, then set $\uparrow a = \uparrow A$. Let Q be a subset of B. If Q is a countable, then Q is almost ω^{op}-like by Lemma 2.4. Therefore, we may assume that $\kappa > \omega$ and the theorem is true for all free boolean algebras of size less than κ.

We will construct a continuous elementary chain $\langle M_\alpha \rangle_{\alpha < \kappa}$ of elementary submodels of H_θ and a continuous increasing sequence of sets $\langle D_\alpha \rangle_{\alpha < \kappa}$ satisfying the following conditions for all $\alpha < \kappa$.

(1) $\alpha \cup \{B, \wedge, \vee, Q\} \subseteq M_\alpha$ and $|M_\alpha| \leq |\alpha| + \omega$.
(2) D_α is a dense subset of $Q \cap M_\alpha$.
(3) $D_\alpha \cap \uparrow q$ is finite for all $q \in Q \cap M_\alpha$.
(4) $D_{\alpha+1} \cap \uparrow q = D_\alpha \cap \uparrow q$ for all $q \in Q \cap M_\alpha$.

Given this construction, set $D = \bigcup_{\alpha < \kappa} D_\alpha$. Then D is a dense subset of Q by (2). Moreover, if $\alpha < \kappa$ and $d \in D_\alpha$, then $d \in Q \cap M_\alpha$ by (2); whence, d is below at most finitely many elements of D by (3) and (4). Hence, Q is almost ω^{op}-like.

For stage 0, choose any $M_0 \prec H_\theta$ satisfying (1). Since $Q \cap M_0 \subseteq B \cap M_0$, we may choose D_0 to be an ω^{op}-like dense subset of $Q \cap M_0$, exactly what (2) and (3) require. At limit stages, (1) and (2) are clearly preserved, and (3) is preserved because of (4).

For a successor stage $\alpha + 1$, choose $M_{\alpha+1}$ such that $M_\alpha \prec M_{\alpha+1} \prec H_\theta$ and (1) holds for stage $\alpha+1$. Since $Q \cap M_{\alpha+1} \subseteq B \cap M_{\alpha+1}$, there is an ω^{op}-like dense subset E of $Q \cap M_{\alpha+1}$. Set $D_{\alpha+1} = D_\alpha \cup (E \setminus {\uparrow}(Q \cap M_\alpha))$. Then (4) is easily verified: if $q \in Q \cap M_\alpha$, then

$$D_{\alpha+1} \cap {\uparrow}q = (D_\alpha \cap {\uparrow}q) \cup ((E \cap {\uparrow}q) \setminus {\uparrow}(Q \cap M_\alpha)) = D_\alpha \cap {\uparrow}q.$$

Let us verify (2) for stage $\alpha + 1$. Let $q \in Q \cap M_{\alpha+1}$. If $q \in {\uparrow}(Q \cap M_\alpha)$, then $q \in {\uparrow}D_\alpha \subseteq {\uparrow}D_{\alpha+1}$ because of (2) for stage α. Suppose $q \notin {\uparrow}(Q \cap M_\alpha)$. Choose $e \in E$ such that $e \leq q$. Then $e \notin {\uparrow}(Q \cap M_\alpha)$; hence, $q \in {\uparrow}(E \setminus {\uparrow}(Q \cap M_\alpha)) \subseteq {\uparrow}D_{\alpha+1}$.

It remains only to verify (3) for stage $\alpha + 1$. Let $q \in Q \cap M_{\alpha+1}$. Then $E \cap {\uparrow}q$ is finite; hence, by the definition of $D_{\alpha+1}$, it suffices to show that $D_\alpha \cap {\uparrow}q$ is finite. By Lemma 3.1, there exists $r \in B \cap M_\alpha$ such that $r \geq q$ and $M_\alpha \cap {\uparrow}q = M_\alpha \cap {\uparrow}r$; hence, $D_\alpha \cap {\uparrow}q = D_\alpha \cap {\uparrow}r$. Since $q \in Q$, we have $r \in M_\alpha \cap {\uparrow}Q$. By elementarity, there exists $p \in Q \cap M_\alpha$ such that $p \leq r$; hence, $D_\alpha \cap {\uparrow}r \subseteq D_\alpha \cap {\uparrow}p$. By (2) for stage α, we have $D_\alpha \cap {\uparrow}p$ is finite; hence, $D_\alpha \cap {\uparrow}q$ is finite. $\square$

Definition 3.3. For any space X, let $\mathrm{Clop}(X)$ denote the boolean algebra of clopen subsets of X.

Theorem 3.4. *Let X be a dyadic compactum and let $\mathcal{U}$ be a family of subsets of X such that for all $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $\overline{V} \cap \overline{X \setminus U} = \emptyset$. Then $\mathcal{U}$ is almost ω^{op}-like.*

Proof. Let $f \colon 2^\kappa \to X$ be a continuous surjection for some cardinal κ. Set $\mathcal{B} = \mathrm{Clop}(2^\kappa)$. Then $\mathcal{B}$ is a free boolean algebra. Set $\mathcal{V} = \{f^{-1}U : U \in \mathcal{U}\}$. Then it suffices to show that $\mathcal{V}$ is almost ω^{op}-like. Let $\mathcal{Q}$ denote the set of all $B \in \mathcal{B}$ such that $V \subseteq B$ for some $V \in \mathcal{V}$. By Theorem 3.2, $\mathcal{Q}$ is almost ω^{op}-like. Hence, by Lemma 2.21, it suffices to show that $\mathcal{Q}$ and $\mathcal{V}$ are mutually dense. By definition, every $Q \in \mathcal{Q}$ contains some $V \in \mathcal{V}$; hence, it suffices to show that every $V \in \mathcal{V}$ contains some $Q \in \mathcal{Q}$. Suppose $V \in \mathcal{V}$. Choose $U \in \mathcal{U}$ such that $\overline{U} \cap \overline{X \setminus f[V]} = \emptyset$. Then there exists $B \in \mathcal{B}$ such that $f^{-1}\overline{U} \subseteq B \subseteq V$; hence, $V \supseteq B \in \mathcal{Q}$. $\square$

The following corollary is immediate and it implies the first half of Theorem 1.5.

Corollary 3.5. *Let X be a dyadic compactum. Then, for all closed subsets C of X, every neighborhood base of C contains an ω^{op}-like neighborhood base of C. Moreover, every π-base of X contains an ω^{op}-like π-base of X.*

Remark. The first half of the above corollary can also be proved simply by citing Theorem 2.19 and Lemma 2.21.

Next we state the natural generalizations of Lemma 3.1, Theorem 3.2, Theorem 3.4, and Corollary 3.5 to continuous images of products of compacta with bounded weight. We will only remark briefly about the proofs of these generalizations, for they are easy modifications of the corresponding old proofs.

Lemma 3.6. *Let κ be a regular uncountable cardinal and let B be a coproduct $\coprod_{i \in I} B_i$ of boolean algebras all of size less than κ; let $\{B, \wedge, \vee, \langle B_i \rangle_{i \in I}\} \subseteq M \prec H_\theta$ and $M \cap \kappa \in \kappa + 1$. Then, for all $q \in B$, there exists $r \in B \cap M$ such that, for all $p \in B \cap M$, we have $p \geq q$ if and only if $p \geq r$. In particular, $r \geq q$.*

Proof. Note that the subalgebra $B \cap M$ is the subcoproduct $\coprod_{i \in I \cap M} B_i$ naturally embedded in B. Then proceed as in the proof of Lemma 3.1 with $\bigcup_{i \in I} B_i$, naturally embedded in B, playing the role of G. $\square$

Theorem 3.7. *Let $\kappa \geq \omega$ and B be a coproduct of boolean algebras all of size at most κ. Then every subset of B is almost κ^{op}-like.*

Proof. The proof is essentially the proof of Theorem 3.2. Instead of using Lemma 3.1, use the instance of Lemma 3.6 for the regular uncountable cardinal κ^+. $\square$

Theorem 3.8. *Let $\kappa \geq \omega$ and let X be Hausdorff and a continuous image of a product of compacta all of weight at most κ; let $\mathcal{U}$ be a family of subsets of X such that, for all $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $\overline{V} \cap \overline{X \setminus U} = \emptyset$. Then $\mathcal{U}$ is almost κ^{op}-like.*

Proof. Let $h \colon \prod_{i \in I} X_i \to X$ be a continuous surjection where each X_i is a compactum with weight at most κ. Each X_i embeds into $[0,1]^\kappa$ and is therefore a continuous image of a closed subspace of 2^κ. Hence, we may assume $\prod_{i \in I} X_i$ is totally disconnected. The rest of the proof is just the proof of Theorem 3.4 with Theorem 3.7 replacing Theorem 3.2. $\square$

The following corollary is immediate.

Corollary 3.9. *Let $\kappa \geq \omega$ and let X be Hausdorff and a continuous image of a product of compacta all of weight at most κ. Then, for all closed subsets C of X, every neighborhood base of C contains a κ^{op}-like neighborhood base of C. Moreover, every π-base of X contains a κ^{op}-like π-base of X.*

Remark. Again, the first half of the above corollary can also proved simply by citing Theorem 2.19 and Lemma 2.21.

In contrast to Corollary 3.5, not all dyadic compacta have ω^{op}-like bases. The following proposition is essentially due to Peregudov (see Lemma 1 of [23]). It makes it easy to produces examples of dyadic compacta X such that $Nt(X) > \omega$.

Proposition 3.10. *Suppose a point p in a space X satisfies $\pi\chi(p,X) < \mathrm{cf}\,\kappa = \kappa \leq \chi(p,X)$. Then $Nt(X) > \kappa$.*

Proof. Let $\mathcal{A}$ be a base of X. Let $\mathcal{U}_0$ and $\mathcal{V}_0$ be, respectively, a local π-base at p of size at most $\pi\chi(p,X)$ and a local base at p of size $\chi(p,X)$. For each element of $\mathcal{U}_0$, choose a subset in $\mathcal{A}$, thereby producing a local π-base $\mathcal{U}$ at p that is a subset of $\mathcal{A}$ of size at most $\pi\chi(p,X)$. Similarly, for each element of $\mathcal{V}_0$, choose a smaller neighborhood of p in $\mathcal{A}$, thereby producing a local base $\mathcal{V}$ at p that is a subset of $\mathcal{A}$ of size $\chi(p,X)$. Every element of $\mathcal{V}$ contains an element of $\mathcal{U}$. Hence, some element of $\mathcal{U}$ is contained in κ-many elements of $\mathcal{V}$; hence, $\mathcal{A}$ is not κ^{op}-like. $\square$

Example 3.11. Let X be the discrete sum of 2^ω and 2^{ω_1}. Let Y be the quotient of X resulting from collapsing a point in 2^ω and a point in 2^{ω_1} to a single point p. Then $\pi\chi(p,Y) = \omega$ and $\chi(p,Y) = \omega_1$; hence, $Nt(Y) > \omega_1$.

Question 3.12. Is there a dyadic compactum X such that $\pi\chi(p,X) = \chi(p,X)$ for all $p \in X$ but X has no ω^{op}-like base? In particular, if Y is as in Example 3.11 and Z is the discrete sum of Y and 2^{ω_2}, then does Z^{ω_1} have an ω^{op}-like base?

As we shall see in Theorem 3.21, if we make an additional assumption about a dyadic compactum X, namely, that all its points have π-character equal to its weight, then X has an ω^{op}-like base. Also, we may choose this ω^{op}-like base to be a subset of an arbitrary base of X. To prove this, we approximate such an X by metric compacta. Each such metric compactum is constructed using the following technique due to Bandlow [2].

Definition 3.13. Given a space X, let $C(X)$ denote the set of continuous maps from X to $\mathbb{R}$.

Definition 3.14. Suppose X is a space and $\mathcal{F}$ is a set. For all $p \in X$, let $p/\mathcal{F}$ denote the set of $q \in X$ satisfying $f(p) = f(q)$ for all $f \in \mathcal{F} \cap C(X)$. For each $f \in \mathcal{F}$, define $f/\mathcal{F} \colon X/\mathcal{F} \to \mathbb{R}$ by $(f/\mathcal{F})(p/\mathcal{F}) = f(p)$ for all $p \in X$.

Lemma 3.15. *Suppose X is a compactum and $\mathcal{F} \subseteq C(X)$. Then $X/\mathcal{F}$ (with the quotient topology) is a compactum and its topology is the coarsest topology for which $f/\mathcal{F}$ is continuous for all $f \in \mathcal{F}$. Further suppose $\{X \setminus f^{-1}\{0\} : f \in \mathcal{F}\}$ is a base of X and $\mathcal{F} \in M \prec H_\theta$. Then $\{(X \setminus f^{-1}\{0\})/(\mathcal{F} \cap M) : f \in \mathcal{F} \cap M\}$ is a base of $X/(\mathcal{F} \cap M)$.*

Proof. If $f \in \mathcal{F}$, then $f/\mathcal{F}$ is clearly continuous with respect to the quotient topology of $X/\mathcal{F}$. Therefore, the compact quotient topology on $X/\mathcal{F}$ is finer than the Hausdorff topology induced by $\{f/\mathcal{F} : f \in \mathcal{F}\}$. If a compact topology $\mathcal{T}_0$ is finer than a Hausdorff topology $\mathcal{T}_1$, then $\mathcal{T}_0 = \mathcal{T}_1$. Hence, the quotient topology on $X/\mathcal{F}$ is the topology induced by $\{f/\mathcal{F} : f \in \mathcal{F}\}$.

Set $\mathcal{A} = \{X \setminus f^{-1}\{0\} : f \in \mathcal{F}\}$. Suppose $\mathcal{A}$ is a base of X and $\mathcal{F} \in M \prec H_\theta$. Let us show that $\{(X \setminus f^{-1}\{0\})/(\mathcal{F} \cap M) : f \in \mathcal{F} \cap M\}$ is a base of $X/(\mathcal{F} \cap M)$. Let $\mathcal{U}$ denote the set of preimages of open rational intervals with respect to elements of $\mathcal{F} \cap M$. Let $\mathcal{V}$ denote the set of nonempty finite intersections of elements of $\mathcal{U}$. Then $\mathcal{V} \subseteq M$ and $\{V/(\mathcal{F} \cap M) : V \in \mathcal{V}\}$ is base of $X/(\mathcal{F} \cap M)$. Suppose $p \in V_0 \in \mathcal{V}$. Then it suffices to find $W \in \mathcal{A} \cap M$ such that $p \in W \subseteq V_0$. Choose $V_1 \in \mathcal{V}$ such that $p \in V_1 \subseteq \overline{V}_1 \subseteq V_0$. Then there exist $n < \omega$ and $W_0, \dots, W_{n-1} \in \mathcal{A}$ such that $\overline{V}_1 \subseteq \bigcup_{i<n} W_i \subseteq V_0$. By elementarity, we may assume $W_0, \dots, W_{n-1} \in M$. Hence, there exists $i < n$ such that $p \in W_i \subseteq V_0$ and $W_i \in \mathcal{A} \cap M$. $\square$

To construct an ω^{op}-like base of a suitable dyadic compactum X, we apply Lemma 2.9 to a family of spaces $X/(\mathcal{F} \cap M)$ where $\mathcal{F} \subseteq C(X)$ and M ranges over a transfinite sequence of countable elementary submodels of H_θ. This sequence is constructed such that, loosely speaking, each submodel in the sequence knows about the preceding submodels.

Definition 3.16. Let κ be a regular uncountable cardinal and let $\langle H_\theta, \dots \rangle$ be an expansion of the $\{\in\}$-structure H_θ to an $\mathcal{L}$-structure for some language $\mathcal{L}$ of size less than κ. Then a *κ-approximation sequence* in $\langle H_\theta, \dots \rangle$ is an ordinally indexed sequence $\langle M_\alpha \rangle_{\alpha < \eta}$ such that for all $\alpha < \eta$ we have $\{\kappa, \langle M_\beta \rangle_{\beta < \alpha}\} \subseteq M_\alpha \prec \langle H_\theta, \dots \rangle$ and $|M_\alpha| \subseteq M_\alpha \cap \kappa \in \kappa$.

The following lemma is a generalization of a technique of Jackson and Mauldin [13] of approximating a structure by a tree of elementary substructures.

Lemma 3.17. *Let κ and $\langle H_\theta, \dots \rangle$ be as in Definition 3.16. Then there is a $\{\kappa\}$-definable map Ψ that sends every κ-approximation sequence $\langle M_\alpha \rangle_{\alpha < \eta}$ in $\langle H_\theta, \dots \rangle$ to a sequence $\langle \Sigma_\alpha \rangle_{\alpha \leq \eta}$ such that we have the following for all $\alpha \leq \eta$.*

(1) Σ_α *is a finite set.*
(2) $|N| \subseteq N \prec \langle H_\theta, \ldots \rangle$ *for all* $N \in \Sigma_\alpha$.
(3) $\bigcup \Sigma_\alpha = \bigcup_{\beta < \alpha} M_\beta$.
(4) *If* $\alpha < \eta$, *then* $\Sigma_\alpha \in M_\alpha$.
(5) Σ_α *is an* $\in$-*chain.*
(6) *If* $N_0, N_1 \in \Sigma_\alpha$ *and* $N_0 \in N_1$, *then* $|N_0| > |N_1|$.
(7) $\langle \Sigma_\beta \rangle_{\beta \leq \alpha} = \Psi(\langle M_\beta \rangle_{\beta < \alpha})$.

Moreover, $|\Sigma_\lambda| = 1$ *and* $\{\alpha < \lambda : |\Sigma_\alpha| = 1\}$ *is closed unbounded in* λ *for all infinite cardinals* $\lambda \leq \eta$.

Proof. Let Ω denote the class of $\langle \gamma_i \rangle_{i<n} \in \mathrm{On}^{<\omega} \setminus \{\emptyset\}$ for which $\kappa \leq |\gamma_i| > |\gamma_j|$ for all $i < j < n$ and $|\gamma_{n-1}| < \kappa$. Order Ω lexicographically and let Υ be the order isomorphism from On to Ω. Given any $\sigma = \langle \gamma_i \rangle_{i<n} \in \mathrm{On}^{<\omega}$ and $i < n$, set $\phi_i(\sigma) = \langle \gamma_0, \ldots, \gamma_{i-1}, 0 \rangle$ and $\phi_n(\sigma) = \sigma$. Let $\langle M_\alpha \rangle_{\alpha < \eta}$ be a κ-approximation sequence in $\langle H_\theta, \ldots \rangle$. For all $\alpha \leq \eta$ and $i \in \mathrm{dom}\,\Upsilon(\alpha)$, set

$$N_{\alpha,i} = \bigcup \{M_\beta : \phi_i(\Upsilon(\alpha)) \leq \Upsilon(\beta) < \phi_{i+1}(\Upsilon(\alpha))\};$$

set $\Sigma_\alpha = \{N_{\alpha,i} : i \in \mathrm{dom}\,\Upsilon(\alpha)\} \setminus \{\emptyset\}$. Then Ψ is $\{\kappa\}$-definable and it is easily verified that $|\Sigma_\lambda| = 1$ and $\{\alpha < \lambda : |\Sigma_\alpha| = 1\}$ is closed unbounded in λ for all infinite cardinals $\lambda \leq \eta$. Let us prove (1)-(7). (1), (3), (4), and (7) immediately follow from the relevant definitions. Let $\alpha \leq \eta$ and $\langle \beta_i \rangle_{i<n} = \Upsilon(\alpha)$. We may assume $n > 0$. For all $\sigma \in \Omega$ and $i < n - 1$, we have $\phi_i(\Upsilon(\alpha)) \leq \sigma < \phi_{i+1}(\Upsilon(\alpha))$ if and only if σ is the concatenation of $\langle \beta_j \rangle_{j<i}$ and some $\tau \in \Omega$ satisfying $\tau < \langle \beta_i, 0 \rangle$. Therefore, $|N_{\alpha,i}| = |\beta_i|$ for all $i < n - 1$. For all $\sigma \in \Omega$, we have $\phi_{n-1}(\Upsilon(\alpha)) \leq \sigma < \phi_n(\Upsilon(\alpha))$ if and only if $\sigma = \langle \beta_0, \ldots, \beta_{n-2}, \gamma \rangle$ for some $\gamma < \beta_{n-1}$. Hence, $|N_{\alpha,n-1}| < \kappa$; hence, $|N_{\alpha,i}| > |N_{\alpha,j}|$ for all $i < j < n$. Let $\Upsilon(\alpha_i) = \phi_i(\Upsilon(\alpha))$ for all $i < n$. If $i < j < n$, then $\{N_{\alpha,k} : k < j\} = \Sigma_{\alpha_{j-1}}$; whence, either $N_{\alpha,j} = \emptyset$ or $N_{\alpha,i} \in M_{\alpha_{j-1}} \subseteq N_{\alpha,j}$, depending on whether $\beta_j = 0$. Thus, (5) and (6) hold.

Finally, let us prove (2). Proceed by induction on α. Suppose $\beta_{n-1} > 0$. Since $\{N_{\alpha,i} : i < n-1\} = \Sigma_{\alpha_{n-1}}$ and $\alpha_{n-1} + \beta_{n-1} = \alpha$, it suffices to show that $|N_{\alpha,n-1}| \subseteq N_{\alpha,n-1} \prec \langle H_\theta, \ldots \rangle$. If $\beta_{n-1} \in \mathrm{Lim}$, then $N_{\alpha,n-1}$ is the union of the $\in$-chain $\langle N_{\alpha_{n-1}+\gamma,n-1} \rangle_{\gamma < \beta_{n-1}}$; hence, $|N_{\alpha,n-1}| \subseteq N_{\alpha,n-1} \prec \langle H_\theta, \ldots \rangle$. If $\beta_{n-1} \notin \mathrm{Lim}$, then $N_{\alpha,n-1} = N_{\alpha-1,n-1} \cup M_{\alpha-1} = M_{\alpha-1}$ because $N_{\alpha-1,n-1} \in M_{\alpha-1}$ and $|N_{\alpha-1,n-1}| < \kappa$; hence, $|N_{\alpha,n-1}| \subseteq N_{\alpha,n-1} \prec \langle H_\theta, \ldots \rangle$.

Therefore, we may assume $\beta_{n-1} = 0$. Hence, $\Sigma_\alpha = \{N_{\alpha,i} : i < n-1\}$; hence, we may assume $n > 1$. Since $\{N_{\alpha,i} : i < n-2\} = \Sigma_{\alpha_{n-2}}$ and $\alpha_{n-2} < \alpha$, it suffices to show that $|N_{\alpha,n-2}| \subseteq N_{\alpha,n-2} \prec \langle H_\theta, \ldots \rangle$. If $\beta_{n-2} = \kappa$, then $N_{\alpha,n-2}$ is the union of the $\in$-chain $\langle N_{\alpha_{n-2}+\gamma,n-2} \rangle_{\gamma < \kappa}$; hence, $|N_{\alpha,n-2}| \subseteq N_{\alpha,n-2} \prec \langle H_\theta, \ldots \rangle$. Hence, we may assume $\beta_{n-2} > \kappa$. Let $\Upsilon(\delta_\gamma) = \langle \beta_0, \ldots, \beta_{n-3}, \gamma, 0 \rangle$ for all $\gamma \in [\kappa, \beta_{n-2})$. If $\beta_{n-2} \in \mathrm{Lim}$, then $N_{\alpha,n-2}$ is the union of the $\in$-chain $\langle N_{\delta_\gamma,n-2} \rangle_{\kappa \leq \gamma < \beta_{n-2}}$; hence, $|N_{\alpha,n-2}| \subseteq N_{\alpha,n-2} \prec \langle H_\theta, \ldots \rangle$. Hence, we may let $\beta_{n-2} = \varepsilon + 1$. Suppose $|\varepsilon| = \kappa$. Then $N_{\alpha,n-2} = N_{\delta_\varepsilon,n-2} \cup \bigcup_{\gamma < \kappa} M_{\delta_\varepsilon+\gamma}$. If $\gamma < \kappa$, then $\phi_{n-1}(\Upsilon(\delta_\varepsilon + \gamma)) = \Upsilon(\delta_\varepsilon)$; whence, δ_ε and γ are definable from $\delta_\varepsilon + \gamma$ and κ; whence, $\gamma \cup \bigcup_{\rho < \gamma} M_{\delta_\varepsilon+\rho} \subseteq M_{\delta_\varepsilon+\gamma}$. Hence, $|N_{\delta_\varepsilon,n-2}| = \kappa \subseteq \bigcup_{\gamma < \kappa} M_{\delta_\varepsilon+\gamma} \prec \langle H_\theta, \ldots \rangle$. Moreover, since $N_{\delta_\varepsilon,n-2} \in M_{\delta_\varepsilon}$, we have $N_{\delta_\varepsilon,n-2} \subseteq \bigcup_{\gamma < \kappa} M_{\delta_\varepsilon+\gamma}$; hence, $|N_{\alpha,n-2}| = \kappa \subseteq N_{\alpha,n-2} \prec \langle H_\theta, \ldots \rangle$.

Therefore, we may assume $|\varepsilon| > \kappa$. Let $\Upsilon(\zeta_\gamma) = \langle \beta_0, \ldots, \beta_{n-3}, \varepsilon, \kappa + \gamma, 0 \rangle$ for all $\gamma < |\varepsilon|$. Then $N_{\alpha,n-2} = N_{\delta_\varepsilon,n-2} \cup \bigcup_{\gamma < |\varepsilon|} N_{\zeta_\gamma,n-1}$. If $\gamma < |\varepsilon|$, then $\Upsilon(\zeta_\gamma)(n-1) = \kappa + \gamma$; whence, $\gamma \in M_{\zeta_\gamma} \subseteq N_{\zeta_{\gamma+1},n-1}$. Hence, $|\varepsilon| \subseteq \bigcup_{\gamma < |\varepsilon|} N_{\zeta_\gamma,n-1} \prec$

$\langle H_\theta, \ldots \rangle$. Since $|N_{\delta_\varepsilon,n-2}| = |\varepsilon|$ and $N_{\delta_\varepsilon,n-2} \in M_{\delta_\varepsilon} \subseteq N_{\zeta_0,n-1}$, we have $N_{\delta_\varepsilon,n-2} \subseteq \bigcup_{\gamma<|\varepsilon|} N_{\zeta_\gamma,n-1}$. Hence, $|N_{\alpha,n-2}| = |\varepsilon| \subseteq N_{\alpha,n-2} \prec \langle H_\theta, \ldots \rangle$. □

Proposition 3.18. *If X is a topological space, then every base of X contains a base of size at most $w(X)$.*

Proof. Let $\mathcal{A}$ be an arbitrary base of X; let $\mathcal{B}$ be a base of X of size at most $w(X)$. Since X is hereditarily $w(X)^+$-compact, we may choose, for each $U \in \mathcal{B}$, some $\mathcal{A}_U \in [\mathcal{A}]^{\leq w(X)}$ such that $U = \bigcup \mathcal{A}_U$. Then $\bigcup \{\mathcal{A}_U : U \in \mathcal{B}\}$ is a base of X and in $[\mathcal{A}]^{\leq w(X)}$. □

Lemma 3.19. *Let X be a dyadic compactum such that $\pi\chi(p,X) = w(X)$ for all $p \in X$. Let $\mathcal{A}$ be a base of X consisting only of cozero sets. Then $\mathcal{A}$ contains an ω^{op}-like base of X.*

Proof. Set $\kappa = w(X)$; by Proposition 3.18, we may assume $|\mathcal{A}| = \kappa$. Choose $\mathcal{F} \subseteq C(X)$ such that $\mathcal{A} = \{X \setminus g^{-1}\{0\} : g \in \mathcal{F}\}$. Let $h: 2^\lambda \to X$ be a continuous surjection for some cardinal λ. Let $\mathcal{B}$ be the free boolean algebra $\mathrm{Clop}(2^\lambda)$. By Lemma 2.9, we may assume $\kappa > \omega$. Let $\langle M_\alpha \rangle_{\alpha<\kappa}$ be an ω_1-approximation sequence in $\langle H_\theta, \in, \mathcal{F}, h \rangle$; set $\langle \Sigma_\alpha \rangle_{\alpha\leq\kappa} = \Psi(\langle M_\alpha \rangle_{\alpha<\kappa})$ as defined in Lemma 3.17.

For each $\alpha < \kappa$, set $\mathcal{A}_\alpha = \mathcal{A} \cap M_\alpha$ and $\mathcal{F}_\alpha = \mathcal{F} \cap M_\alpha$. For every $\mathcal{H} \subseteq \mathcal{A}_\alpha$, let $\mathcal{H}/\mathcal{F}_\alpha$ denote $\{U/\mathcal{F}_\alpha : U \in \mathcal{H}\}$. By Lemma 3.15, $\mathcal{A}_\alpha/\mathcal{F}_\alpha$ is a base of $X/\mathcal{F}_\alpha$. Since $X/\mathcal{F}_\alpha$ is a metric compactum, there exists $\mathcal{W}_\alpha \subseteq \mathcal{A}_\alpha$ such that $\mathcal{W}_\alpha/\mathcal{F}_\alpha$ is a base of $X/\mathcal{F}_\alpha$ satisfying (2), (3), and (4) of Lemma 2.9. By (2) of Lemma 2.9, we may choose, for each $U \in \mathcal{W}_\alpha$, some $E_{\alpha,U} \in \mathcal{B} \cap M_\alpha$ such that $h^{-1}\overline{U} \subseteq E_{\alpha,U} \subseteq h^{-1}V$ for all $V \in \mathcal{W}_\alpha$ satisfying $\overline{U} \subseteq V$. Set $\mathcal{G}_\alpha = \{E_{\alpha,U} : U \in \mathcal{W}_\alpha\}$.

Suppose $\mathcal{G}_\alpha$ is not ω^{op}-like. Then there exist $U \in \mathcal{W}_\alpha$ and $\langle V_n \rangle_{n<\omega} \in \mathcal{W}_\alpha^\omega$ such that $E_{\alpha,U} \subsetneq E_{\alpha,V_n} \neq E_{\alpha,V_m}$ for all $m < n < \omega$. Set $\Gamma = \{W \in \mathcal{W}_\alpha : U \subsetneq W\}$. By (2) of Lemma 2.9, Γ is finite; hence, by (4) of Lemma 2.9, there exists $n < \omega$ such that $\{W \in \mathcal{W}_\alpha : V_n \subsetneq W\} \not\subseteq \Gamma$. Hence, there exists $W \in \mathcal{W}_\alpha$ such that W strictly contains V_n but not U. Hence, by (3) of Lemma 2.9, $E_{\alpha,V_n} \subseteq h^{-1}W$; hence, $h^{-1}U \subseteq E_{\alpha,U} \subsetneq E_{\alpha,V_n} \subseteq h^{-1}W$; hence, $U \subsetneq W$, which is absurd. Therefore, $\mathcal{G}_\alpha$ is ω^{op}-like.

Let $\mathcal{V}_\alpha$ denote the set of $V \in \mathcal{W}_\alpha$ satisfying $U \not\subseteq V$ for all nonempty open $U \in \bigcup \Sigma_\alpha$. Let us show that $\mathcal{V}_\alpha/\mathcal{F}_\alpha$ is a base of $X/\mathcal{F}_\alpha$. If $V \in \mathcal{V}_\alpha$, then $\mathcal{P}(V) \cap \mathcal{W}_\alpha \subseteq \mathcal{V}_\alpha$; hence, it suffices to show that $\mathcal{V}_\alpha$ covers X. Since $|\bigcup \Sigma_\alpha| < \kappa$, every point of X has a neighborhood in $\mathcal{A}$ that does not contain any nonempty open subset of X in $\bigcup \Sigma_\alpha$. By compactness, there is cover of X by finitely many such neighborhoods, say, $W_0, \ldots, W_{n-1}$. By elementarity, we may assume $W_0, \ldots, W_{n-1} \in \mathcal{A}_\alpha$. Then $\{W_i : i < n\}$ has a refining cover $\mathcal{S} \subseteq \mathcal{W}_\alpha$. Hence, $\mathcal{S} \subseteq \mathcal{V}_\alpha$; hence, $\mathcal{V}_\alpha$ covers X as desired.

Let $\mathcal{U}_\alpha$ denote the set of $U \in \mathcal{V}_\alpha$ such that $\overline{U} \subseteq V$ for some $V \in \mathcal{V}_\alpha$. Then $\mathcal{U}_\alpha/\mathcal{F}_\alpha$ is clearly a base of $X/\mathcal{F}_\alpha$. Set $\mathcal{E}_\alpha = \{E_{\alpha,U} : U \in \mathcal{U}_\alpha\}$. Then $\mathcal{E}_\alpha$ is ω^{op}-like because it is a subset of $\mathcal{G}_\alpha$.

For all $\mathcal{I} \subseteq \mathcal{P}(2^\kappa)$, set $\uparrow\mathcal{I} = \{H \subseteq 2^\kappa : H \supseteq I \text{ for some } I \in \mathcal{I}\}$. For all $H \subseteq 2^\kappa$, set $\uparrow H = \uparrow\{H\}$. Set $\mathcal{U} = \bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$ and $\mathcal{C} = \mathcal{B} \cap \uparrow\{h^{-1}U : U \in \mathcal{U}\}$. For all $\alpha \leq \kappa$, set $\mathcal{D}_\alpha = \bigcup_{\beta<\alpha} \mathcal{E}_\beta$. Then we claim the following for all $\alpha \leq \kappa$.

(1) $\mathcal{D}_\alpha$ is a dense subset of $\mathcal{C} \cap \bigcup \Sigma_\alpha$.
(2) $\mathcal{D}_\alpha \cap \uparrow H$ is finite for all $H \in \mathcal{C} \cap \bigcup \Sigma_\alpha$.
(3) If $\alpha < \kappa$, then $\mathcal{D}_{\alpha+1} \cap \uparrow H = \mathcal{D}_\alpha \cap \uparrow H$ for all $H \in \mathcal{C} \cap \bigcup \Sigma_\alpha$.

We prove this claim by induction. For stage 0, the claim is vacuous. For limit stages, (1) is clearly preserved, and (2) is preserved because of (3). Suppose $\alpha < \kappa$ and (1) and (2) hold for stage α. Then it suffices to prove (3) for stage α and to prove (1) and (2) for stage $\alpha + 1$.

Let us verify (3). Seeking a contradiction, suppose $H \in \mathcal{C} \cap \bigcup \Sigma_\alpha$ and $\mathcal{D}_{\alpha+1} \cap {\uparrow}H \neq \mathcal{D}_\alpha \cap {\uparrow}H$. Then $\mathcal{E}_\alpha \cap {\uparrow}H \neq \emptyset$; hence, there exists $U \in \mathcal{U}_\alpha$ such that $H \subseteq E_{\alpha,U}$. By (1), there exist $\beta < \alpha$ and $W \in \mathcal{U}_\beta$ such that $E_{\beta,W} \subseteq H$. By definition, there exists $V \in \mathcal{V}_\alpha$ such that $\overline{U} \subseteq V$. Hence, $h^{-1}W \subseteq E_{\beta,W} \subseteq H \subseteq E_{\alpha,U} \subseteq h^{-1}V$; hence, $W \subseteq V$. Since $W \in M_\beta \subseteq \bigcup \Sigma_\alpha$ and $V \in \mathcal{V}_\alpha$, we have $W \nsubseteq V$, which yields our desired contradiction.

Let us verify (1) for stage $\alpha + 1$. By (1) for stage α, we have

$$\mathcal{D}_{\alpha+1} = \mathcal{D}_\alpha \cup \mathcal{E}_\alpha \subseteq \left(\mathcal{C} \cap \bigcup \Sigma_\alpha \right) \cup (\mathcal{C} \cap M_\alpha) = \mathcal{C} \cap \bigcup \Sigma_{\alpha+1},$$

so we just need to show denseness. Let $H \in \mathcal{C} \cap \bigcup \Sigma_{\alpha+1}$. If $H \in \bigcup \Sigma_\alpha$, then $H \in {\uparrow}\mathcal{D}_\alpha$, so we may assume $H \in M_\alpha$. By elementarity, there exists $U_0 \in \mathcal{U}_\alpha$ such that $h^{-1}U_0 \subseteq H$. Choose $U_1 \in \mathcal{U}_\alpha$ such that $\overline{U}_1 \subseteq U_0$. Then $E_{\alpha,U_1} \subseteq h^{-1}U_0$; hence, $E_{\alpha,U_1} \subseteq H$. Hence, $H \in {\uparrow}\mathcal{D}_{\alpha+1}$.

To complete the proof of the claim, let us verify (2) for stage $\alpha + 1$. By (1) for stage $\alpha + 1$, it suffices to prove $\mathcal{D}_{\alpha+1} \cap {\uparrow}H$ is finite for all $H \in \mathcal{D}_{\alpha+1}$. By (3), if $H \in \mathcal{D}_\alpha$, then $\mathcal{D}_{\alpha+1} \cap {\uparrow}H = \mathcal{D}_\alpha \cap {\uparrow}H$, which is finite by (1) and (2) for stage α. Hence, we may assume $H \in \mathcal{E}_\alpha$. Since $\mathcal{E}_\alpha$ is ω^{op}-like, it suffices to show that $\mathcal{D}_\alpha \cap {\uparrow}H$ is finite. Since $\mathcal{D}_\alpha \subseteq \bigcup \Sigma_\alpha$, it suffices to show that $\mathcal{D}_\alpha \cap N \cap {\uparrow}H$ is finite for all $N \in \Sigma_\alpha$. Let $N \in \Sigma_\alpha$. By Lemma 3.1, there exists $G \in \mathcal{B} \cap N$ such that $G \supseteq H$ and $\mathcal{B} \cap N \cap {\uparrow}H = \mathcal{B} \cap N \cap {\uparrow}G$; hence, $\mathcal{D}_\alpha \cap N \cap {\uparrow}H = \mathcal{D}_\alpha \cap N \cap {\uparrow}G$. Since $G \supseteq H \in \mathcal{C}$, we have $G \in \mathcal{C}$. By (2) for stage α, the set $\mathcal{D}_\alpha \cap N \cap {\uparrow}G$ is finite; hence, $\mathcal{D}_\alpha \cap N \cap {\uparrow}H$ is finite.

Since $\mathcal{U} \subseteq \mathcal{A}$, it suffices to prove that $\mathcal{U}$ is an ω^{op}-like base of X. Suppose $p \in V \in \mathcal{A}$. Then there exists $\alpha < \kappa$ such that $V \in \mathcal{A}_\alpha$. Hence, there exists $U \in \mathcal{U}_\alpha$ such that $p/\mathcal{F}_\alpha \in U/\mathcal{F}_\alpha \subseteq V/\mathcal{F}_\alpha$; hence, $p \in U \subseteq V$. Thus, $\mathcal{U}$ is a base of X.

Let us show that $\mathcal{U}$ is ω^{op}-like. Suppose not. Then there exists $\alpha < \kappa$ and $U_0 \in \mathcal{U}_\alpha$ such that there exist infinitely many $V \in \mathcal{U}$ such that $U_0 \subseteq V$. Choose $U_1 \in \mathcal{U}_\alpha$ such that $\overline{U}_1 \subseteq U_0$. Suppose $\beta < \kappa$ and $U_0 \subseteq V \in \mathcal{U}_\beta$. Then $E_{\alpha,U_1} \subseteq h^{-1}U_0 \subseteq h^{-1}V \subseteq E_{\beta,V}$. By (1) and (2), $\mathcal{D}_\kappa$ is ω^{op}-like; hence, there are only finitely many possible values for $E_{\beta,V}$. Therefore, there exist $\langle \gamma_n \rangle_{n<\omega} \in \kappa^\omega$ and $\langle V_n \rangle_{n<\omega} \in \prod_{n<\omega} \mathcal{U}_{\gamma_n}$ such that $V_m \neq V_n$ and $E_{\gamma_m,V_m} = E_{\gamma_n,V_n}$ for all $m < n < \omega$. Suppose that for some $\delta < \kappa$ we have $\gamma_n = \delta$ for all $n < \omega$. Let $i < \omega$ and set $\Gamma = \{W \in \mathcal{W}_\delta : V_i \subsetneq W\}$. By (2) and (4) of Lemma 2.9, there exists $j < \omega$ such that $\{W \in \mathcal{W}_\delta : V_j \subsetneq W\} \nsubseteq \Gamma$. Hence, there exists $W \in \mathcal{W}_\delta$ such that W strictly contains V_j but not V_i. By (3) of Lemma 2.9, $\overline{V}_j \subseteq W$. Hence, $h^{-1}\overline{V}_i \subseteq E_{\delta,V_i} = E_{\delta,V_j} \subseteq h^{-1}W$. Hence, $\overline{V}_i \subseteq W$. Since W does not strictly contain V_i, we must have $V_i = \overline{V}_i = W$. Hence, $h^{-1}V_i = E_{\delta,V_i} = E_{\delta,V_0}$. Since i was arbitrary chosen, we have $V_m = V_n = h[E_{\delta,V_0}]$ for all $m, n < \omega$, which is absurd. Therefore, our supposed δ does not exist; hence, we may assume $\gamma_0 < \gamma_1$. By definition, there exists $W \in \mathcal{V}_{\gamma_1}$ such that $\overline{V}_1 \subseteq W$. Therefore, $h^{-1}V_0 \subseteq E_{\gamma_0,V_0} = E_{\gamma_1,V_1} \subseteq h^{-1}W$; hence, $V_0 \subseteq W$. Since $V_0 \in M_{\gamma_0} \subseteq \bigcup \Sigma_{\gamma_1}$ and $W \in \mathcal{V}_{\gamma_1}$, we have $V_0 \nsubseteq W$, which is absurd. Therefore, $\mathcal{U}$ is ω^{op}-like. $\square$

Let us show that we may remove the requirement that the base $\mathcal{A}$ in Lemma 3.19 consist only of cozero sets.

Lemma 3.20. *Suppose X is a space with no isolated points and $\chi(p, X) = w(X)$ for all $p \in X$. Further suppose $\kappa = \operatorname{cf}\kappa \leq \min\{Nt(X), w(X)\}$ and X has a network consisting of at most $w(X)$-many κ-compact sets. Then every base of X contains an $Nt(X)^{\mathrm{op}}$-like base of X.*

Proof. Set $\lambda = Nt(X)$ and $\mu = w(X)$. Let $\mathcal{A}$ be an arbitrary base of X; let $\mathcal{B}$ be a λ^{op}-like base of X; let $\mathcal{N}$ be a network of X consisting of at most μ-many κ-compact sets. By Proposition 3.18, we may assume $|\mathcal{B}| = \mu$. Let $\langle\langle N_\alpha, B_\alpha\rangle\rangle_{\alpha<\mu}$ enumerate $\{\langle N, B\rangle \in \mathcal{N} \times \mathcal{B} : N \subseteq B\}$. Construct a sequence $\langle \mathcal{G}_\alpha \rangle_{\alpha<\mu}$ as follows. Suppose $\alpha < \mu$ and $\langle \mathcal{G}_\beta \rangle_{\beta<\alpha}$ is a sequence of elements of $[\mathcal{B}]^{<\kappa}$. For each $p \in N_\alpha$, we have $\chi(p, X) = \mu \geq \kappa = \operatorname{cf}\kappa$; hence, we may choose $U_{\alpha,p} \in \mathcal{B}$ such that $p \in U_{\alpha,p} \notin \bigcup_{\beta<\alpha} \mathcal{G}_\beta$. Choose $\sigma_\alpha \in \left[N_\alpha\right]^{<\kappa}$ such that $N_\alpha \subseteq \bigcup_{p\in\sigma_\alpha} U_{\alpha,p}$. Set $\mathcal{G}_\alpha = \{U_{\alpha,p} : p \in \sigma_\alpha\}$.

For each $\alpha < \mu$, choose $\mathcal{F}_\alpha \in [\mathcal{A}]^{<\kappa}$ such that $N_\alpha \subseteq \bigcup\mathcal{F}_\alpha \subseteq B_\alpha$ and $\mathcal{F}_\alpha$ refines $\mathcal{G}_\alpha$. Set $\mathcal{F} = \bigcup_{\alpha<\mu} \mathcal{F}_\alpha$, which is easily seen to be a base of X. Let us show that $\mathcal{F}$ is λ^{op}-like. Suppose not. Then, since $\kappa = \operatorname{cf}\kappa \leq \lambda$, there exist $V \in \mathcal{F}$, $I \in [\mu]^\lambda$, and $\langle W_\alpha \rangle_{\alpha\in I} \in \prod_{\alpha\in I}\mathcal{F}_\alpha$ such that $V \subseteq \bigcap_{\alpha\in I} W_\alpha$. For each $\alpha \in I$, there is a superset of W_α in $\mathcal{G}_\alpha$. By induction, $\mathcal{G}_\alpha \cap \mathcal{G}_\beta = \emptyset$ for all $\alpha < \beta < \mu$; hence, V has λ-many supersets in the λ^{op}-like base $\mathcal{B}$, which is absurd, for V has a subset in $\mathcal{B}$.   □

Remark. If X is regular and locally κ-compact and $\kappa \leq w(X)$, then it is easily seen that X has a network consisting of at most $w(X)$-many κ-compact sets.

Theorem 3.21. *Let X be a dyadic compactum such that $\pi\chi(p, X) = w(X)$ for all $p \in X$. Then every base $\mathcal{A}$ of X contains an ω^{op}-like base of X.*

Proof. By Lemma 3.19, $Nt(X) = \omega$. Since $w(X) = \pi\chi(p, X) \leq \chi(p, X) \leq w(X)$ for all $p \in X$, we may apply Lemma 3.20 to get a subset of $\mathcal{A}$ that is an ω^{op}-like base of X.   □

Finally, let us prove the second half of Theorem 1.5.

Corollary 3.22. *Let X be a homogeneous dyadic compactum with base $\mathcal{A}$. Then $\mathcal{A}$ contains an ω^{op}-like base of X.*

Proof. Efimov [6] and Gerlits [9] independently proved that the π-character of every dyadic compactum is equal to its weight. Since X is homogeneous, $\pi\chi(p, X) = w(X)$ for all $p \in X$. Hence, $\mathcal{A}$ contains an ω^{op}-like base of X by Theorem 3.21.   □

Note that a compactum is dyadic if and only if it a continous image of a product of second countable compacta. Let us prove generalizations of Theorem 3.21 and Corollary 3.22 about continuous images of products of compacta with bounded weight.

Lemma 3.23. *Suppose $\kappa = \operatorname{cf}\kappa > \omega$ and X is a space such that $\pi\chi(p, X) = w(X) \geq \kappa$ for all $p \in X$. Further suppose X has a network consisting of at most $w(X)$-many κ-compact closed sets. Then every base of X contains a $w(X)^{\mathrm{op}}$-like base of X.*

Proof. Set $\lambda = w(X)$ and let $\mathcal{A}$ be an arbitrary base of X. By Proposition 3.18, we may assume $|\mathcal{A}| = \lambda$. Let $\mathcal{N}$ be a network of X consisting of at most λ-many κ-compact sets. Let $\langle M_\alpha \rangle_{\alpha < \lambda}$ be a continuous elementary chain such that for all $\alpha < \lambda$ we have $\mathcal{A}, \mathcal{N}, M_\alpha \in M_{\alpha+1} \prec H_\theta$. We may also require that $M_\alpha \cap \kappa \in \kappa > |M_\alpha|$ for all $\alpha < \kappa$ and $|M_\alpha| = |\kappa + \alpha|$ for all $\alpha \in \lambda \setminus \kappa$. For each $\alpha < \lambda$, set $\mathcal{A}_\alpha = \mathcal{A} \cap M_\alpha$. Set $\mathcal{B} = \bigcup_{\alpha < \lambda} \mathcal{A}_{\alpha+1} \setminus \uparrow \mathcal{A}_\alpha$, which is clearly λ^{op}-like. Let us show that $\mathcal{B}$ is a base of X. Suppose $p \in U \in \mathcal{A}$. Choose $N \in \mathcal{N}$ such that $p \in N \subseteq U$. Choose $\alpha < \lambda$ such that $N, U \in \mathcal{A}_{\alpha+1}$. For each $q \in N$, choose $V_q \in \mathcal{A} \setminus \uparrow \mathcal{A}_\alpha$ such that $q \in V_q \subseteq U$. Then there exists $\sigma \in [N]^{<\kappa}$ such that $N \subseteq \bigcup_{q \in \sigma} V_q$. By elementarity, we may assume $\langle V_q \rangle_{q \in \sigma} \in M_{\alpha+1}$. Choose $q \in \sigma$ such that $p \in V_q$. Then $V_q \in \mathcal{B}$ and $p \in V_q \subseteq U$. Thus, $\mathcal{B}$ is a base of X. $\qquad\square$

Theorem 3.24. *Let $\kappa \geq \omega$ and let X be Hausdorff and a continuous image of a product of compacta each with weight at most κ. Suppose $\pi\chi(p, X) = w(X)$ for all $p \in X$. Then every base of X contains a κ^{op}-like base.*

Proof. Let $h \colon \prod_{i \in I} X_i \to X$ be a continuous surjection where each X_i is a compactum with weight at most κ. Each X_i embeds into $[0,1]^\kappa$ and is therefore a continuous image of a closed subspace of 2^κ. Hence, we may assume $\prod_{i \in I} X_i$ is totally disconnected. Set $\lambda = w(X)$; by Lemmas 2.9 and 3.23, we may assume $\lambda > \kappa$. By Theorem 3.21, we may assume $\kappa > \omega$. Inductively construct a κ^+-approximation sequence $\langle M_\alpha \rangle_{\alpha < \lambda}$ in $\langle H_\theta, \in, C(X), h, \langle \mathrm{Clop}(X_i) \rangle_{i \in I} \rangle$ as follows. For each $\alpha < \lambda$, let $\langle N_{\alpha,\beta} \rangle_{\beta < \kappa}$ be an ω_1-approximation sequence in

$$\langle H_\theta, \in, C(X), h, \kappa, \langle \mathrm{Clop}(X_i) \rangle_{i \in I}, \langle M_\beta \rangle_{\beta < \alpha} \rangle.$$

Set $\langle \Gamma_{\alpha,\beta} \rangle_{\beta \leq \kappa} = \Psi(\langle N_{\alpha,\beta} \rangle_{\beta < \kappa})$ as defined in Lemma 3.17; let $\{M_\alpha\} = \Gamma_{\alpha,\kappa}$. Set $\langle \Sigma_\alpha \rangle_{\alpha \leq \lambda} = \Psi(\langle M_\alpha \rangle_{\alpha < \lambda})$. Set $\mathcal{F} = C(X) \cap \bigcup \Sigma_\lambda$ and $\mathcal{A} = \{X \setminus f^{-1}\{0\} : f \in \mathcal{F}\}$. Then $\mathcal{A}$ is a base of X. By Lemma 3.20, it suffices to construct a subset of $\mathcal{A}$ that is a κ^{op}-like base of X.

For each $\alpha < \lambda$, set $\mathcal{F}_\alpha = \mathcal{F} \cap M_\alpha$. Let $\mathcal{V}_\alpha$ denote the set of $V \in \mathcal{A} \cap M_\alpha$ satisfying $U \not\subseteq V$ for all nonempty open $U \in \bigcup \Sigma_\alpha$. Arguing as in the proof Lemma 3.19, $\mathcal{V}_\alpha / \mathcal{F}_\alpha$ is a base of $X / \mathcal{F}_\alpha$. For each $\beta < \kappa$, let $\mathcal{V}_{\alpha,\beta}$ denote the set of all $V \in \mathcal{V}_\alpha \cap N_{\alpha,\beta}$ satisfying $U \not\subseteq V$ for all nonempty open $U \in \bigcup \Gamma_{\alpha,\beta}$. Let $\mathcal{R}_{\alpha,\beta}$ denote the set of $\langle U, V \rangle \in \mathcal{V}_{\alpha,\beta}^2$ for which $\overline{U} \subseteq V$; set $\mathcal{U}_{\alpha,\beta} = \mathrm{dom}\, \mathcal{R}_{\alpha,\beta}$; set $\mathcal{U}_\alpha = \bigcup_{\beta < \kappa} \mathcal{U}_{\alpha,\beta}$.

Let us show that $\mathcal{U}_\alpha / \mathcal{F}_\alpha$ is also a base of $X / \mathcal{F}_\alpha$. Suppose $p \in V \in \mathcal{V}_\alpha$. Extend $\{V\}$ to a finite subcover σ of $\mathcal{V}_\alpha$ such that $p \notin \bigcup(\sigma \setminus \{V\})$. Choose $\beta < \kappa$ such that $\sigma \in N_{\alpha,\beta}$. For each $q \in X$, choose $V_{q,0}, V_{q,1} \in \mathcal{A}$ such that $q \in V_{q,0}$ and there exists $W \in \sigma$ such that $U \not\subseteq \overline{V}_{q,0} \subseteq V_{q,1} \subseteq W$ for all nonempty open $U \in \bigcup \Sigma_\alpha \cup \bigcup \Gamma_{\alpha,\beta}$. Choose $\tau \in [X]^{<\omega}$ such that $X = \bigcup_{q \in \tau} V_{q,0}$. By elementarity, we may assume $\langle V_{q,i} \rangle_{\langle q,i \rangle \in \tau \times 2} \in N_{\alpha,\beta}$. Choose $q \in \tau$ such that $p \in V_{q,0}$. Then $V_{q,0} \in \mathcal{U}_{\alpha,\beta}$ and $p \in V_{q,0} \subseteq V$. Thus, $\mathcal{U}_\alpha / \mathcal{F}_\alpha$ is a base of $X / \mathcal{F}_\alpha$.

Set $\mathcal{B} = \mathrm{Clop}\left(\prod_{i \in I} X_i\right)$. For each $\langle U_0, U_1 \rangle \in \bigcup_{\beta < \kappa} \mathcal{R}_{\alpha,\beta}$, choose $E_\alpha(U_0, U_1) \in \mathcal{B} \cap M_\alpha$ such that $h^{-1}\overline{U}_0 \subseteq E_\alpha(U_0, U_1) \subseteq h^{-1}U_1$. Set $\mathcal{E}_{\alpha,\beta} = E_\alpha[\mathcal{R}_{\alpha,\beta}]$. Set $\mathcal{E}_\alpha = \bigcup_{\beta < \kappa} \mathcal{E}_{\alpha,\beta}$. Let us show that $\mathcal{E}_\alpha$ is κ^{op}-like. Suppose $\beta, \gamma < \kappa$ and $\mathcal{E}_{\alpha,\beta} \ni H \subseteq K \in \mathcal{E}_{\alpha,\gamma}$. Then it suffices to show that $\gamma \leq \beta$. Seeking a contradiction, suppose $\beta < \gamma$. There exist $\langle U_0, U_1 \rangle \in \mathcal{R}_{\alpha,\beta}$ and $\langle V_0, V_1 \rangle \in \mathcal{R}_{\alpha,\gamma}$ such that $H = E_\alpha(U_0, U_1)$ and $K = E_\alpha(V_0, V_1)$. Hence, $\bigcup \Gamma_{\alpha,\gamma} \ni U_0 \subseteq V_1 \in \mathcal{V}_{\alpha,\gamma}$, in contradiction with the definition of $\mathcal{V}_{\alpha,\gamma}$.

Set $\mathcal{U} = \bigcup_{\alpha < \lambda} \mathcal{U}_\alpha$ and $\mathcal{C} = \mathcal{B} \cap \mathop{\uparrow}\{h^{-1}U : U \in \mathcal{U}\}$. For all $\alpha \leq \lambda$, set $\mathcal{D}_\alpha = \bigcup_{\beta < \alpha} \mathcal{E}_\beta$. Then we claim the following for all $\alpha \leq \lambda$.

  (1) $\mathcal{D}_\alpha$ is a dense subset of $\mathcal{C} \cap \bigcup \Sigma_\alpha$.
  (2) $|\mathcal{D}_\alpha \cap \mathop{\uparrow} H| < \kappa$ for all $H \in \mathcal{C} \cap \bigcup \Sigma_\alpha$.
  (3) If $\alpha < \lambda$, then $\mathcal{D}_{\alpha+1} \cap \mathop{\uparrow} H = \mathcal{D}_\alpha \cap \mathop{\uparrow} H$ for all $H \in \mathcal{C} \cap \bigcup \Sigma_\alpha$.

We prove this claim by induction. For stage 0, the claim is vacuous. For limit stages, (1) is clearly preserved, and (2) is preserved because of (3). Suppose $\alpha < \kappa$ and (1) and (2) hold for stage α. Then it suffices to prove (3) for stage α and to prove (1) and (2) for stage $\alpha + 1$.

Let us verify (3). Seeking a contradiction, suppose $H \in \mathcal{C} \cap \bigcup \Sigma_\alpha$ and $\mathcal{D}_{\alpha+1} \cap \mathop{\uparrow} H \neq \mathcal{D}_\alpha \cap \mathop{\uparrow} H$. Then $\mathcal{E}_\alpha \cap \mathop{\uparrow} H \neq \emptyset$; hence, there exists $V \in \mathcal{U}_\alpha$ such that $H \subseteq h^{-1}V$. By (1), there exist $\beta < \alpha$ and $U \in \mathcal{U}_\beta$ and $K \in \mathcal{E}_\beta$ such that $h^{-1}\overline{U} \subseteq K \subseteq H$. Hence, $U \subseteq V$. Since $U \in M_\beta \subseteq \bigcup \Sigma_\alpha$ and $V \in \mathcal{V}_\alpha$, we have $U \not\subseteq V$, which yields our desired contradiction.

Let us verify (1) for stage $\alpha + 1$. By (1) for stage α, we have

$$\mathcal{D}_{\alpha+1} = \mathcal{D}_\alpha \cup \mathcal{E}_\alpha \subseteq \left(\mathcal{C} \cap \bigcup \Sigma_\alpha \right) \cup (\mathcal{C} \cap M_\alpha) = \mathcal{C} \cap \bigcup \Sigma_{\alpha+1},$$

so we just need to show denseness. Let $H \in \mathcal{C} \cap \bigcup \Sigma_{\alpha+1}$. If $H \in \bigcup \Sigma_\alpha$, then $H \in \mathop{\uparrow} \mathcal{D}_\alpha$, so we may assume $H \in M_\alpha$. By elementarity, there exists $U \in \mathcal{U}_\alpha$ such that $h^{-1}U \subseteq H$. Choose $\beta < \kappa$ such that $U \in \mathcal{U}_{\alpha,\beta}$; choose $V \in \mathcal{U}_{\alpha,\beta}$ such that $\overline{V} \subseteq U$. Then $E_\alpha(V, U) \subseteq H$; hence, $H \in \mathop{\uparrow} \mathcal{D}_{\alpha+1}$.

The proof of the claim is completed by noting that (2) for stage $\alpha + 1$ can be verfied just as in the proof of Lemma 3.19, except that Lemma 3.6 is used in place of Lemma 3.1.

Just as in the proof of Lemma 3.19, $\mathcal{U}$ is a base of X; hence, it suffices to show that $\mathcal{U}$ is κ^{op}-like. Suppose $\gamma < \lambda$ and $\delta < \kappa$ and $U \in \mathcal{U}_{\gamma,\delta}$ and $\langle \langle \zeta_\alpha, \eta_\alpha \rangle \rangle_{\alpha < \kappa} \in (\lambda \times \kappa)^\kappa$ and $\langle W_\alpha \rangle_{\alpha < \kappa} \in \prod_{\alpha < \kappa} \mathcal{U}_{\zeta_\alpha, \eta_\alpha}$ and $U \subseteq \bigcap_{\alpha < \kappa} W_\alpha$. Then it suffices to show that $W_\alpha = W_\beta$ for some $\alpha < \beta < \kappa$. Choose $V \in \mathcal{U}_{\gamma,\delta}$ such that $\overline{V} \subseteq U$. For each $\alpha < \kappa$, choose $V_\alpha \in \mathcal{V}_{\zeta_\alpha, \eta_\alpha}$ such that $\overline{W}_\alpha \subseteq V_\alpha$; set $H_\alpha = E_{\zeta_\alpha}(W_\alpha, V_\alpha)$. Then $E_\gamma(V, U) \subseteq \bigcap_{\alpha < \kappa} H_\alpha$. By (1) and (2), $\mathcal{D}_\lambda$ is κ^{op}-like; hence, there exists $J \in [\kappa]^{\omega_1}$ such that $H_\alpha = H_\beta$ for all $\alpha, \beta \in J$; hence, $W_\alpha \subseteq V_\beta$ for all $\alpha, \beta \in J$. If $\alpha, \beta \in J$ and $\zeta_\alpha < \zeta_\beta$, then $\bigcup \Sigma_{\zeta_\beta} \ni W_\alpha \subseteq V_\beta$, in contradiction with $V_\beta \in \mathcal{V}_{\zeta_\beta}$. Hence, $\zeta_\alpha = \zeta_\beta$ for all $\alpha, \beta \in J$. If $\alpha, \beta \in J$ and $\eta_\alpha < \eta_\beta$, then $\bigcup \Gamma_{\zeta_\beta, \eta_\beta} \ni W_\alpha \subseteq V_\beta$, in contradiction with $V_\beta \in \mathcal{V}_{\zeta_\beta, \eta_\beta}$. Hence, $\eta_\alpha = \eta_\beta$ for all $\alpha, \beta \in J$. Hence, $\{W_\alpha : \alpha \in J\} \subseteq N_{\zeta_{\min J}, \eta_{\min J}}$; hence, $W_\alpha = W_\beta$ for some $\alpha < \beta < \kappa$. $\qquad\square$

Lemma 3.25. *Let κ be an uncountable regular cardinal; let X be a compactum such that $w(X) \geq \kappa$ and X is a continuous image of a product of compacta each with weight less than κ. Then $\pi(X) = w(X)$.*

Proof. It suffices to prove that $\pi(X) \geq \kappa$. Seeking a contradiction, suppose $\mathcal{A}$ is a π-base of X of size less than κ. Let $\langle X_i \rangle_{i \in I}$ be a sequence of compacta each with weight less than κ and let h be a continuous surjection from $\prod_{i \in I} X_i$ to X. Choose $M \prec H_\theta$ such that $\mathcal{A} \cup \{C(X), h, \langle C(X_i) \rangle_{i \in I}\} \subseteq M$ and $|M| = |\mathcal{A}|$. Choose $p \in M \cap \prod_{i \in I} X_i$ and set $Y = \{q \in \prod_{i \in I} X_i : p \restriction (I \setminus M) = q \restriction (I \setminus M)\}$. Then it suffices to show that $h[Y] = X$, for that implies $\kappa \leq w(X) \leq w(Y) < \kappa$. Seeking a contradiction, suppose $h[Y] \neq X$. Then there exists $U \in \mathcal{A}$ such that $U \cap h[Y] = \emptyset$. By elementarity, there exists $\sigma \in [I \cap M]^{<\omega}$ and $\langle V_i \rangle_{i \in \sigma}$ such that

V_i is a nonempty open subset of X_i for all $i \in \sigma$, and $\bigcap_{i \in \sigma} \pi_i^{-1} V_i \subseteq h^{-1} U$. Hence, $Y \cap \bigcap_{i \in \sigma} \pi_i^{-1} V_i \neq \emptyset$, in contradiction with $U \cap h[Y] = \emptyset$. $\qquad\square$

Definition 3.26. Given any cardinal κ, set $\log \kappa = \min\{\lambda : 2^\lambda \geq \kappa\}$.

Lemma 3.27. *Let κ be an uncountable regular cardinal; let X be a compactum such that $w(X) \geq \kappa$ and X is a continuous image of a product of compacta each with weight less than κ. Then $\pi\chi(X) = w(X)$.*

Proof. Let $\langle X_i \rangle_{i \in I}$ be a sequence of compacta each with weight less than κ and let h be a continuous surjection from $\prod_{i \in I} X_i$ to X. For any space Y, we have $\pi(Y) = \pi\chi(Y)d(Y)$. Hence, $w(X) = \pi(X) = \pi\chi(X)d(X)$ by Lemma 3.25; hence, we may assume $d(X) = w(X)$. Arguing as in the proof of Lemma 3.25, if $\mathcal{A}$ is a π-base of X and $\mathcal{A} \cup \{C(X), h, \langle C(X_i)\rangle_{i \in I}\} \subseteq M \prec H_\theta$, then X is a continuous image of $\prod_{i \in I \cap M} X_i$; hence, we may assume $|I| = \pi(X)$. By 5.5 of [14], $d(X) \leq d(\prod_{i \in I} X_i) \leq \kappa \cdot \log|I|$. By 2.37 of [14], $d(Y) \leq \pi\chi(Y)^{c(Y)}$ for all T_3 non-discrete spaces Y. Since κ is a caliber of X_i for all $i \in I$, it is also a caliber of X; hence, $|I| = \pi(X) = d(X) \leq \pi\chi(X)^\kappa$; hence, $\log|I| \leq \kappa \cdot \pi\chi(X)$. Therefore, $w(X) = d(X) \leq \kappa \cdot \pi\chi(X)$; hence, we may assume $w(X) = \kappa$.

Let $\langle U_\alpha \rangle_{\alpha < \kappa}$ enumerate a base of X. For each $\alpha < \kappa$, choose $p_\alpha \in U_\alpha$. Since $d(X) = w(X) = \kappa$, there is no $\alpha < \kappa$ such that $\{p_\beta : \beta < \alpha\}$ is dense in X. Since κ is a caliber of X, we may choose $p \in X \setminus \bigcup_{\alpha < \kappa} \overline{\{p_\beta : \beta < \alpha\}}$. It suffices to show that $\pi\chi(p, X) = \kappa$. Seeking a contradiction, suppose $\pi\chi(p, X) < \kappa$. Then there exists $\alpha < \kappa$ such that $\{U_\beta : \beta < \alpha\}$ contains a local π-base at p; hence, $p \in \overline{\{p_\beta : \beta < \alpha\}}$, in contradiction with how we chose p. $\qquad\square$

Theorem 3.28. *Let $\langle X_i \rangle_{i \in I}$ be a sequence of compacta; let X be a homogeneous compactum; let $h\colon \prod_{i \in I} X_i \to X$ be a continuous surjection. If there is a regular cardinal κ such that $w(X_i) < \kappa \leq w(X)$ for all $i \in I$, then every base of X contains a $(\sup_{i \in I} w(X_i))^{\mathrm{op}}$-like base. Otherwise, $w(X) \leq \sup_{i \in I} w(X_i)$ and every base of X trivially contains a $(w(X)^+)^{\mathrm{op}}$-like base.*

Proof. The latter case is a trivial application of Proposition 3.18. In the former case, Lemma 3.27 implies $\pi\chi(p, X) = w(X)$ for all $p \in X$; apply Theorem 3.24. $\quad\square$

Every known homogeneous compactum is a continuous image of a product of compacta each with weight at most $\mathfrak{c}$; hence, Theorem 3.28 provides a uniform justification for our observation that all known homogeneous compacta have Noetherian type at most $\mathfrak{c}^+$. Analogously, since every known homogeneous compactum is such a continuous image, it has $\mathfrak{c}^+$ among its calibers; hence, it has cellularity at most $\mathfrak{c}$.

Let us now turn to the spectrum of Noetherian types of dyadic compacta and a proof of Theorem 1.6.

Theorem 3.29. *Let κ and λ be infinite cardinals such that $\lambda < \kappa$. Let X be the discrete sum of 2^κ and 2^λ. Let Y be the quotient space induced by collapsing $\langle 0 \rangle_{\alpha < \kappa}$ and $\langle 0 \rangle_{\alpha < \lambda}$ to a single point p. If $\lambda < \mathrm{cf}\,\kappa$, then $Nt(Y) = \kappa^+$. If $\lambda \geq \mathrm{cf}\,\kappa$, then $Nt(Y) = \kappa$.*

Proof. Clearly $\chi(p, Y) = \kappa$ and $\pi\chi(p, Y) = \lambda$. Hence, if $\lambda < \mathrm{cf}\,\kappa$, then $\kappa^+ \leq Nt(Y) \leq w(Y)^+ = \kappa^+$ by Proposition 3.10. Suppose $\lambda \geq \mathrm{cf}\,\kappa$. We still have $\kappa \leq Nt(Y)$ by Proposition 3.10, so it suffices to construct a κ^{op}-like base of Y. Let

$\sim$ be the equivalence relation such that $Y = X/\sim$. In building a base of Y, we proceed in the canonical way when away from p: for each $\mu \in \{\kappa, \lambda\}$, set

$$\mathcal{A}_\mu = \{\{x \in 2^\mu : \eta \subseteq x\}/\sim \,:\, \eta \in \mathrm{Fn}(\mu, 2) \text{ and } \eta^{-1}\{1\} \neq \emptyset\}.$$

Choose $f_0 \colon \kappa \to \mathrm{cf}\,\kappa$ such that for all $\alpha < \mathrm{cf}\,\kappa$ the preimage $f_0^{-1}\{\alpha\}$ is bounded in κ. Define $f \colon [\kappa]^{<\omega} \to \mathrm{cf}\,\kappa$ by $f(\sigma) = f_0(\sup \sigma)$ for all $\sigma \in [\kappa]^{<\omega}$. Choose $g_0 \colon \lambda \to \mathrm{cf}\,\kappa$ such that for all $\alpha < \mathrm{cf}\,\kappa$ the preimage $g_0^{-1}\{\alpha\}$ is unbounded in λ. Define $g \colon [\lambda]^{<\omega} \to \mathrm{cf}\,\kappa$ by $g(\sigma) = g_0(\sup \sigma)$ for all $\sigma \in [\lambda]^{<\omega}$. Set

$$\mathcal{A}_p = \bigcup_{\alpha < \mathrm{cf}\,\kappa} \Big\{ \big(\{x \in 2^\kappa : x[\sigma] = \{0\}\} \cup \{x \in 2^\lambda : x[\tau] = \{0\}\}\big)/\sim \,:$$

$$\langle \sigma, \tau \rangle \in f^{-1}\{\alpha\} \times g^{-1}\{\alpha\} \Big\}.$$

Set $\mathcal{A} = \mathcal{A}_\kappa \cup \mathcal{A}_\lambda \cup \mathcal{A}_p$. Let us show that $\mathcal{A}$ is a κ^{op}-like base of Y. The only nontrivial aspect of showing that $\mathcal{A}$ is a base of Y is verifying that $\mathcal{A}_p$ is a local base at p. Suppose U is an open neighborhood of p. Then there exist $\sigma \in [\kappa]^{<\omega}$ and $\tau \in [\lambda]^{<\omega}$ such that

$$\big(\{x \in 2^\kappa : x[\sigma] = \{0\}\} \cup \{x \in 2^\lambda : x[\tau] = \{0\}\}\big)/\sim \,\subseteq U.$$

Choose $\alpha < \lambda$ such that $\sup \tau < \alpha$ and $g_0(\alpha) = f(\sigma)$. Set $\tau' = \tau \cup \{\alpha\}$ and

$$V = \big(\{x \in 2^\kappa : x[\sigma] = \{0\}\} \cup \{x \in 2^\lambda : x[\tau'] = \{0\}\}\big)/\sim \,.$$

Then $V \subseteq U$ and $V \in \mathcal{A}_p$ because $f(\sigma) = g(\tau')$. Thus, $\mathcal{A}$ is a base of Y.

Let us show that $\mathcal{A}$ is κ^{op}-like. Suppose $U, V \in \mathcal{A}$ and $U \subseteq V$. If $U \in \mathcal{A}_\kappa$, then, fixing U, there are only finitely possibilities for V in $\mathcal{A}_\kappa$; the same is true if κ is replaced by λ or p. Hence, we may assume $U \in \mathcal{A}_i$ and $V \in \mathcal{A}_j$ for some $\{i, j\} \in [\{\kappa, \lambda, p\}]^2$. Since no element of $\mathcal{A}_p$ is a subset of an element of $\mathcal{A}_\kappa \cup \mathcal{A}_\lambda$, we have $i \neq p$. Hence, there exists $\eta \in \mathrm{Fn}(i, 2)$ such that $U = \{x \in 2^i : \eta \subseteq x\}/\sim$. Since $\bigcup \mathcal{A}_\kappa \cap \bigcup \mathcal{A}_\lambda = \emptyset$, we have $j = p$. Hence, there exist $\sigma \in [\kappa]^{<\omega}$ and $\tau \in [\lambda]^{<\omega}$ such that

$$V = \big(\{x \in 2^\kappa : x[\sigma] = \{0\}\} \cup \{x \in 2^\lambda : x[\tau] = \{0\}\}\big)/\sim \,.$$

If $i = \kappa$, then $\sigma \subseteq \eta^{-1}\{0\}$; hence, fixing U, there are only finitely many possibilities for σ, and at most λ-many possibilities for τ. If $i = \lambda$, then $\tau \subseteq \eta^{-1}\{0\}$; hence, fixing U, there are only finitely many possibilities for τ, and at most $|\sup f_0^{-1}\{g(\tau)\}|^{<\omega}$-many possibilities for σ given τ. Thus, there are fewer than κ-many possibilities for V given U. Thus, $\mathcal{A}$ is κ^{op}-like. $\square$

Corollary 3.30. *If κ is a cardinal of uncountable cofinality, then there is a totally disconnected dyadic compactum with Noetherian type κ^+. If κ is a singular cardinal, then there is a totally disconnected dyadic compactum with Noetherian type κ.*

Proof. For the first case, apply Theorem 3.29 with $\lambda = \omega$. For the second case, apply Theorem 3.29 with $\lambda = \mathrm{cf}\,\kappa$. $\square$

Combining the above corollary with the following theorem (and a trivial example like $Nt(2^\omega) = \omega$) immediately proves Theorem 1.6.

Theorem 3.31. *Let X be a dyadic compactum with base $\mathcal{A}$ consisting only of cozero sets. If $Nt(X) \leq \omega_1$, then $\mathcal{A}$ contains an ω^{op}-like base of X. Hence, no dyadic compactum has Noetherian type ω_1.*

Proof. Let $\mathcal{Q}$ be an ω_1^{op}-like base of X of size $w(X)$. Import all the notation from the proof of Lemma 3.19 verbatim, except that we require $\langle M_\alpha \rangle_{\alpha < \kappa}$ to be an ω_1-approximation sequence in $\langle H_\theta, \in, \mathcal{F}, h, \mathcal{Q} \rangle$. Then $\mathcal{U}$ is an ω^{op}-like subset of $\mathcal{A}$ as before. On the other hand, $\mathcal{V}_\alpha / \mathcal{F}_\alpha$ is not necessarily a base of $X / \mathcal{F}_\alpha$ for all $\alpha < \kappa$. However, we will show that $\mathcal{U}$ is still a base of X. In doing so, we will repeatedly use the fact that if $U, \mathcal{Q} \in M \prec H_\theta$ and U is a nonempty open subset of X, then all supersets of U in $\mathcal{Q}$ are in M because $\{V \in \mathcal{Q} : U \subseteq V\}$ is a countable element of M.

Suppose $q \in Q \in \mathcal{Q}$. Then it suffices to find $U \in \mathcal{U}$ such that $q \in U \subseteq Q$. Let β be the least $\alpha < \kappa$ such that there exists $A \in \mathcal{A}_\alpha$ satisfying $q \in A \subseteq \overline{A} \subseteq Q$. Fix such an $A \in \mathcal{A}_\beta$. For each $p \in \overline{A}$, choose $\langle A_p, Q_p \rangle \in \mathcal{A} \times \mathcal{Q}$ such that $p \in A_p \subseteq Q_p \subseteq \overline{Q}_p \subseteq Q$. Since $M_\beta \ni A \subseteq Q \in \mathcal{Q}$, we have $Q \in M_\beta$. Hence, by elementarity, we may assume there exists $\sigma \in \left[\overline{A} \right]^{<\omega}$ such that $\langle \langle A_p, Q_p \rangle \rangle_{p \in \sigma} \in M_\beta$ and $\overline{A} \subseteq \bigcup_{p \in \sigma} A_p$. Choose $p \in \sigma$ such that $q \in A_p$. Suppose $Q_p \notin \bigcup \Sigma_\beta$. Then all nonempty open subsets of Q_p are also not in $\bigcup \Sigma_\beta$; hence, there exist $U \in \mathcal{U}_\beta$ and $V \in \mathcal{V}_\beta$ such that $q / \mathcal{F}_\beta \subseteq U \subseteq V \subseteq A_p \subseteq Q$. Therefore, we may assume $Q_p \in \bigcup \Sigma_\beta$.

Choose $\alpha < \beta$ such that $Q_p \in M_\alpha$. Then $Q \in M_\alpha$ because $Q_p \subseteq Q$. Hence, there exists $\tau \in [\mathcal{A}_\alpha]^{<\omega}$ such that $\overline{Q}_p \subseteq \bigcup \tau \subseteq \overline{\bigcup \tau} \subseteq Q$. Choose $W \in \tau$ such that $q \in W$. Then $q \in W \subseteq \overline{W} \subseteq Q$, in contradiction with the minimality of β. Thus, $\mathcal{U}$ is a base of X. $\qquad\square$

Question 3.32. If κ is an singular cardinal with cofinality ω, then is there a dyadic compactum with Noetherian type κ^+? Is there a dyadic compactum with weakly inaccessible Noetherian type?

We note that the spectrum of Noetherian types of all compacta is trivial.

Theorem 3.33. *Let κ be a regular uncountable cardinal. Then there exists a totally disconnected compactum X such that $Nt(X) = \kappa$ and X has a P_κ-point.*

Proof. Let X be the closed subspace of 2^κ consisting of all $f \in 2^\kappa$ for which $f(\alpha) = 0$ or $f[\alpha] = \{1\}$ for all odd $\alpha < \kappa$. First, let us show that X has a κ^{op}-like base. For each $\sigma \in \mathrm{Fn}(\kappa, 2)$, set $U_\sigma = \{f \in X : f \supseteq \sigma\}$. Let E denote the set of $\sigma \in \mathrm{Fn}(\kappa, 2)$ for which $\sup \mathrm{dom}\,\sigma$ is even and $U_\sigma \neq \emptyset$. Set $\mathcal{A} = \{U_\sigma : \sigma \in E\}$, which is clearly a base of X. Let us show that $\mathcal{A}$ is κ^{op}-like. Suppose $\sigma, \tau \in E$ and $U_\sigma \subseteq U_\tau$. If $\sup \mathrm{dom}\,\sigma < \sup \mathrm{dom}\,\tau$, then for each $f \in U_\sigma$ the sequence

$$ (f \upharpoonright \sup \mathrm{dom}\,\tau) \cup \{ \langle \sup \mathrm{dom}\,\tau, 1 - \tau(\sup \mathrm{dom}\,\tau) \rangle \} \cup \{ \langle \beta, 0 \rangle : \sup \mathrm{dom}\,\tau < \beta < \kappa \} $$

is in $U_\sigma \setminus U_\tau$, which is absurd. Hence, $\sup \mathrm{dom}\,\tau \leq \sup \mathrm{dom}\,\sigma$; hence, there are fewer than κ-many possibilities for τ given σ. Thus, $\mathcal{A}$ is κ^{op}-like.

Finally, it suffices to show that $\langle 1 \rangle_{\alpha < \kappa}$ is a P_κ-point of X, for a P_κ-point must have local Noetherian type at least κ. For each $\alpha < \kappa$, set $\sigma_\alpha = \{\langle 2\alpha + 1, 1 \rangle\}$. Then $\{U_{\sigma_\alpha} : \alpha < \kappa\}$ is a local base at $\langle 1 \rangle_{\alpha < \kappa}$. Moreover, $U_{\sigma_\alpha} \supsetneq U_{\sigma_\beta}$ for all $\alpha < \beta < \kappa$. Since κ is regular, it follows that $\langle 1 \rangle_{\alpha < \kappa}$ is a P_κ-point. $\qquad\square$

Corollary 3.34. *Every infinite cardinal is the Noetherian type of some totally disconnected compactum.*

Proof. By Lemma 2.9, all totally disconnected metric compacta have Noetherian type ω. By Theorem 3.33, if κ is a regular uncountable cardinal, then there is

a totally disconnected compactum X with Noetherian type κ. If κ is a singular cardinal, then there is a totally disconnected dyadic compactum with Noetherian type κ by Corollary 3.30. $\qquad\square$

4. κ-ADIC COMPACTA

The results of the previous section used reflection properties of free boolean algebras—see Lemma 3.1—and more generally coproducts of boolean algebras of bounded size—see Lemma 3.6. Let us define a more general family of reflection properties.

Definition 4.1. Let B be a boolean algebra and let κ and λ be cardinals. Then we say B has the (κ, λ)-FN if and only if, for every M such that $\{B, \wedge, \vee\} \subseteq M \prec H_\theta$ and $|M| \cap \kappa \subseteq M \cap \kappa \in \kappa + 1$, and for every $b \in B$, there exists $A \in [B \cap M]^{<\lambda}$ such that $M \cap \uparrow b = M \cap \uparrow A$.

Remark. For regular κ, the (κ, κ)-FN and the (κ^+, κ)-FN are both equivalent to the κ-FN as defined by Fuchino, Koppelberg, and Shelah [8]. In particular, the (ω_1, ω)-FN is equivalent to the Freese-Nation property and the (ω_2, ω_1)-FN is equivalent to the weak Freese-Nation property.

The (κ, ω)-FN is equivalent to the $(\kappa, 2)$-FN for all κ: if $A \in [B \cap M]^{<\omega}$ and $M \cap \uparrow b = M \cap \uparrow A$, then $\bigwedge A \in M$ and $M \cap \uparrow b = M \cap \uparrow \bigwedge A$. Therefore, a boolean algebra has the (ω_1, ω)-FN if and only if it satisfies the conclusion of Lemma 3.1. Likewise, a boolean algebra satisfies the conclusion of Lemma 3.6 if and only if it has the (κ, ω)-FN.

Theorem 4.2. *If $\kappa \geq \omega$ and B has the $(\kappa^+, \operatorname{cf} \kappa)$-FN, then every subset of B is almost κ^{op}-like.*

Proof. Proceed as in the proof of Theorem 3.2. The only modifications worth noting happen in the last paragraph. Where Lemma 3.1 is used to produce $r \in B \cap M_\alpha$ such that $M_\alpha \cap \uparrow q = M_\alpha \cap \uparrow r$, instead use the $(\kappa^+, \operatorname{cf} \kappa)$-FN to produce $A \in [B \cap M_\alpha]^{<\operatorname{cf} \kappa}$ such that $M_\alpha \cap \uparrow q = M_\alpha \cap \uparrow A$. For each $r \in A$, argue as before that there exists $p_r \in Q \cap M_\alpha$ such that $D_\alpha \cap \uparrow r \subseteq D_\alpha \cap \uparrow p_r$. By an induction hypothesis, $|D_\alpha \cap \uparrow p_r| < \kappa$; hence, $|D_\alpha \cap \uparrow q| \leq |\bigcup_{r \in A} (D_\alpha \cap \uparrow p_r)| < \kappa$. $\qquad\square$

Corollary 4.3. *It is independent of $\neg CH$ whether every separable compactum X satisfies $\chi Nt(X) \leq \omega_1$.*

Proof. Fuchino, Koppelberg, and Shelah [8] proved that $\mathcal{P}(\omega)$ has the (ω_2, ω_1)-FN in the Cohen model. Arguing as in the proof of Theorem 3.4, every separable compactum X, being a continuous image of $\beta\omega$, satisfies $\chi_K Nt(X) \leq \omega_1$ and $\pi Nt(X) \leq \omega_1$ in this model. On the other hand, $\mathfrak{p} = \mathfrak{c}$ implies there is a $P_\mathfrak{c}$-point p in $\beta\omega \setminus \omega$. Assuming $\mathfrak{p} = \mathfrak{c} > \omega_1$, let us show that this p does not have an ω_1^{op}-like base in the separable compactum $\beta\omega$. Let $\mathcal{U}$ be a local base at p in $\beta\omega$. Choose $\mathcal{V} \in [\mathcal{U}]^{\omega_1}$ and $U \in \mathcal{U}$ such that $\overline{U} \setminus \omega \subseteq \bigcap \mathcal{V}$. For every $V \in \mathcal{V}$, the compact set $\overline{U} \setminus V$ is contained in ω, so $\overline{U} \setminus V \subseteq n$ for some $n < \omega$. Therefore, there exist $\mathcal{W} \in [\mathcal{V}]^{\omega_1}$ and $n < \omega$ such that $\overline{U} \setminus W \subseteq n$ for all $W \in \mathcal{W}$. Choose $U_0 \in \mathcal{U}$ such that $U_0 \subseteq U \setminus n$. Then $U_0 \subseteq \bigcap \mathcal{W}$; hence, $\mathcal{U}$ is not ω_1^{op}-like. $\qquad\square$

Theorem 4.4. *Let $\kappa \geq \omega$ and let X be a compactum such that $\pi\chi(p, X) = w(X)$ for all $p \in X$ and such that X is a continuous image of a totally disconnected compactum Y such that $\mathrm{Clop}(Y)$ has the $(\kappa^+, \mathrm{cf}\,\kappa)$-FN. Then every base of X contains an κ^{op}-like base of X.*

Proof. Proceed as in the proof of Theorem 3.24. Modify that proof just as the proof of Theorem 3.2 was modified in the above proof of Theorem 4.2.   □

Ščepin discovered a nice characterization of the Stone spaces of boolean algebras having the (ω_1, ω)-FN.

Definition 4.5 (Ščepin [27])**.** Given a space X, let $RC(X)$ denote the set of regular closed subsets of X. A space X is *k-metrizable* if there exists $\rho\colon X \times RC(X) \to [0, \infty)$ such that we have the following for all $C \in RC(X)$.

   (1) $C = \{x \in X : \rho(x, C) = 0\}$.
   (2) If $C \supseteq B \in RC(X)$, then $\rho(x, C) \leq \rho(x, B)$ for all $x \in X$.
   (3) The map $\rho_C\colon X \to \mathbb{R}$ defined by $\rho_C(x) = \rho(x, C)$ is continuous.
   (4) For each increasing union $\bigcup_{\alpha<\beta} C_\alpha$ of regular closed sets, if $C = \overline{\bigcup_{\alpha<\beta} C_\alpha}$, then $\rho(x, C) = \inf_{\alpha<\beta} \rho(x, C_\alpha)$.

A compactum is *k-adic* if it is a continuous image of k-metrizable compactum.

Remark. Ščepin's notation is "κ-metrizable." Let us use "k-metrizable" for two reasons. First, "κ" has nothing to do with a cardinal κ; it's a Russian abbreviation for canonical. (Canonically closed means regular closed in this context.) Second, for some authors, κ-metrizable means something else, such as having a decreasing uniform base of the form $\{U_\alpha\}_{\alpha<\kappa}$.

The following theorem is implicit in results of Ščepin [27] and more explicit in Heindorf and Šapiro [11]. (See especially Section 2.9 of the latter.)

Theorem 4.6. *A totally disconnected compactum X is k-metrizable if and only if $\mathrm{Clop}(X)$ has the (ω_1, ω)-FN.*

Lemma 4.7 (Ščepin [27])**.** *If X is a k-adic compactum, then $\pi\chi(X) = w(X)$.*

Given the above lemma and the preceding three theorems, it is trivial to generalize our main results from the previous section about the class of dyadic compacta, which are continuous images of powers of 2, to the class of compacta that are continuous images of totally disconnected k-metrizable compacta. Moreover, the next two theorems show that the latter class properly contains the former class.

Theorem 4.8 (Ščepin [27])**.** *Metrizable spaces are k-metrizable. Moreover, products and hyperspaces (with the Vietoris topology) preserve k-metrizability. In particular, every power of 2 is k-metrizable.*

Theorem 4.9 (Šapiro [25])**.** *If $\kappa \geq \omega_2$, then the hyperspace of 2^κ is not dyadic. Hence, there is a totally disconnected compactum that is k-metrizable but not dyadic.*

With a little more care, we can further generalize our results about dyadic compacta to all k-adic compacta.

Definition 4.10. Given a space X and a set M, define $\pi_M^X\colon X \to X/M$ by $\pi_M^X(p) = p/M$.

Lemma 4.11. *Let X be a compactum. Then X is k-metrizable if and only if π_M^X is an open map for all M satisfying $C(X) \in M \prec H_\theta$.*

Proof. Ščepin [28] proved that a compactum X is k-metrizable if and only if, for all sufficiently large regular cardinals μ, there is a closed unbounded $C \subseteq [H_\mu]^\omega$ such that $C(X) \in M \prec H_\mu$ and π_M^X is open for all $M \in C$. (Ščepin stated this result in terms of σ-complete inverse systems of metric compacta; the above formulation is due to Bandlow [3].) It follows at once that X is k-metrizable if π_M^X is open for all M satisfying $C(X) \in M \prec H_\theta$. Conversely, suppose X is k-metrizable and $C(X) \in M \prec H_\theta$. Fix μ and C as above. We may assume $\theta > \mu^\omega$; hence, by elementarity, we may assume $C \in M$. Choose a countable $N \prec H_{(2^{<\theta})^+}$ such that $C(X), C, M \in N$. Then $M \cap N \cap H_\mu \in C$, so $\pi_{M \cap N \cap H_\mu}^X$, which is equal to $\pi_{M \cap N}^X$, is open. Suppose $U \subseteq X$ is open and $p \in U$. Since $\pi_{M \cap N}^X$ is open, there exists a cozero $V \subseteq X$ such that $p \in V \in M \cap N$ and $V/(M \cap N) \subseteq U/(M \cap N)$. The last relation is equivalent to the statement that, for all $q \in V$, there exists $r \in U$ such that, for all $f \in C(X) \cap M \cap N$, we have $f(q) = f(r)$. By elementarity, for every open $U \subseteq X$ and $p \in U$, there exists a cozero $V \subseteq X$ such that $p \in V \in M$ and, for all $q \in V$, there exists $r \in U$ such that, for all $f \in C(X) \cap M$, we have $f(q) = f(r)$. Thus, $p/M \in V/M \subseteq U/M$. Since V is cozero and $V \in M$, the set V/M is cozero. Hence, π_M^X is open. $\square$

Theorem 4.12. *Let X be a k-metrizable compactum and Q a family of cozero subsets of X such that for every $U \in Q$ there exists $V \in Q$ such that $\overline{V} \subseteq U$. Then Q is almost ω^{op}-like.*

Proof. Proceed by induction on $|Q|$. Argue as in the proof of Theorem 3.2 until the verification of (3) for stage $\alpha + 1$, where we need a different argument to show that $D_\alpha \cap {\uparrow}q$ is finite. Let $U = q$ and choose $V \in Q$ such that $\overline{V} \subseteq U$. By Lemma 4.11, U/M_α is open; hence, there exists $f \in C(X) \cap M_\alpha$ such that $V/M_\alpha \subseteq (f^{-1}\{0\})/M_\alpha \subseteq U/M_\alpha$. Since $f \in M_\alpha$, we have $V \subseteq f^{-1}\{0\}$. By elementarity, there exists $W \in Q \cap M_\alpha$ such that $W \subseteq f^{-1}\{0\}$. By (3) for stage α, it suffices to show that $D_\alpha \cap {\uparrow}U \subseteq D_\alpha \cap {\uparrow}W$. Suppose $Z \in D_\alpha \cap {\uparrow}U$. Then $W/M_\alpha \subseteq (f^{-1}\{0\})/M_\alpha \subseteq U/M_\alpha \subseteq Z/M_\alpha$. Since $Z \in D_\alpha \subseteq M_\alpha$ and Z is cozero, we have $W \subseteq Z$. Thus, $D_\alpha \cap {\uparrow}U \subseteq D_\alpha \cap {\uparrow}W$. $\square$

Corollary 4.13. *Let X be a k-adic compactum and $\mathcal{U}$ be a family of subsets of X such that for all $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $\overline{V} \cap \overline{X \setminus U} = \emptyset$. Then $\mathcal{U}$ is almost ω^{op}-like. Hence, $\pi Nt(X) = \chi_K Nt(X) = \omega$.*

Proof. Proceed as in the proof of Theorem 3.4. Use the above theorem instead of Theorem 3.2. $\square$

Theorem 4.14. *Let X be a homogeneous k-adic compactum with base $\mathcal{A}$. Then $\mathcal{A}$ contains an ω^{op}-like base of X.*

Proof. By homogeneity and Lemma 4.7, we have $\pi\chi(p, X) = w(X)$ for all $p \in X$. By Lemma 3.20, we may assume $\mathcal{A}$ consists only of cozero sets. Proceed as in the proof of Lemma 3.19. Replace 2^λ with a k-metrizable compactum Y and replace $\mathcal{B}$ with the set of cozero subsets of Y. For the proof of (2) for stage $\alpha + 1$, we need a different argument that, given $H \in \mathcal{E}_\alpha$ and $N \in \Sigma_\alpha$, the set $\mathcal{D}_\alpha \cap N \cap {\uparrow}H$ is finite.

Choose $U \in \mathcal{U}_\alpha$ such that $H = E_{\alpha, U}$; choose $V \in \mathcal{U}_\alpha$ such that $\overline{V} \subseteq U$. Since π_N^Y is open by Lemma 4.11, we have $(h^{-1}V)/N \subseteq (f^{-1}\{0\})/N \subseteq (h^{-1}U)/N$ for some

$f \in C(Y) \cap N$. Since $f \in N$, we have $h^{-1}V \subseteq f^{-1}\{0\}$. Choose $\beta < \alpha$ such that $f \in M_\beta$. By elementarity, we may choose $W_0 \in \mathcal{A}_\beta$ such that $h^{-1}W_0 \subseteq f^{-1}\{0\}$. Choose $W_1 \in \mathcal{V}_\beta$ such that $\overline{W_1} \subseteq W_0$; choose $W_2 \in \mathcal{U}_\beta$ such that $\overline{W_2} \subseteq W_1$. By (2) for stage α, it suffices to prove $\mathcal{D}_\alpha \cap N \cap \uparrow E_{\alpha,U} \subseteq \uparrow E_{\beta,W_2}$. Suppose $G \in \mathcal{D}_\alpha \cap N \cap \uparrow E_{\alpha,U}$. Then we have

$$(f^{-1}\{0\})/N \subseteq (h^{-1}U)/N \subseteq E_{\alpha,U}/N \subseteq G/N.$$

Since $G \in N$ and G is cozero, we have $f^{-1}\{0\} \subseteq G$. Hence,

$$E_{\beta,W_2} \subseteq h^{-1}W_1 \subseteq h^{-1}W_0 \subseteq f^{-1}\{0\} \subseteq G.$$

Thus, $\mathcal{D}_\alpha \cap N \cap \uparrow E_{\alpha,U} \subseteq \uparrow E_{\beta,W_2}$ as desired. $\qquad\square$

Theorem 4.15. *Let X be a k-adic compactum. Then $Nt(X) \neq \omega_1$.*

Proof. Proceed as in the proof of Theorem 3.31. $\qquad\square$

Question 4.16. Is every k-adic compactum a continuous image of a totally disconnected k-metrizable compactum?

If still greater generality is desired, then one can easily combine the techniques of the proofs of Theorems 4.2, 4.12, and 4.14 to prove the following.

Theorem 4.17. *Let κ be an infinite cardinal and let Y be a compactum such that, for all open $U \subseteq Y$ and for all M satisfying $C(Y) \in M \prec H_\theta$ and $\kappa^+ \cap |M| \subseteq \kappa^+ \cap M \in \kappa^+ + 1$, the set U/M is the intersection of fewer than $(\mathrm{cf}\,\kappa)$-many open subsets of Y/M. If X is Hausdorff and a continuous image of Y, then we have the following.*

(1) *If $\mathcal{U} \subseteq \mathcal{P}(X)$ and, for all $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $\overline{V} \cap \overline{X \setminus U} = \emptyset$, then $\mathcal{U}$ is almost κ^{op}-like. Hence, $\pi Nt(X) \leq \kappa$ and $\chi_K Nt(X) \leq \kappa$.*
(2) *If $\pi\chi(p,X) = w(X)$ for all $p \in X$, then every base of X contains a κ^{op}-like base.*

On the other hand, Lemma 4.7 cannot be so easily generalized. For example, if X is the Stone space of the interval algebra generated by $\{[a,b) : a,b \in \mathbb{R}\}$, then $w(X) = \mathfrak{c}$ and $\pi\chi(X) = \pi(X) = \omega$, despite it being shown in [8] that $\mathrm{Clop}(X)$ has the (ω_2, ω_1)-FN.

5. More on local Noetherian type

In this section, we find two sufficient conditions for a compactum to have a point with an ω^{op}-like local base. The first of these conditions will be used to prove Theorem 1.7. We also present some related results about local bases in terms of Tukey reducibility.

Definition 5.1. Given cardinals $\lambda \geq \kappa \geq \omega$ and a subset E in a space X, a *local $\langle\lambda,\kappa\rangle$-splitter* at E is a set $\mathcal{U}$ of λ-many open neighborhoods of E such that E is not contained in the interior of $\bigcap \mathcal{V}$ for any $\mathcal{V} \in [\mathcal{U}]^\kappa$. If $p \in X$, then we call a local $\langle\lambda,\kappa\rangle$-splitter at $\{p\}$ a local $\langle\lambda,\kappa\rangle$-splitter at p.

Theorem 5.2. *Suppose X is a compactum and $\omega_1 \leq \kappa = \min_{p \in X} \pi\chi(p,X)$. Then there is a local $\langle\kappa,\omega\rangle$-splitter at some $p \in X$.*

Proof. Given any map f, let $\prod f$ denote $\{\langle x_i\rangle_{i\in\operatorname{dom}f} : \forall i \in \operatorname{dom}f\ \ x_i \in f(i)\}$. Given any infinite open family $\mathcal{E}$, let $\Phi(\mathcal{E})$ denote the set of $\langle\sigma,\Gamma\rangle \in [\mathcal{E}]^{<\omega}\times([\mathcal{E}]^{\omega})^{<\omega}$ for which every $\tau \in \prod\Gamma$ satisfies $\bigcap\sigma \subseteq \overline{\bigcup\operatorname{ran}\tau}$. Then $\Phi(\mathcal{E}) = \emptyset$ always implies $\mathcal{E}$ is ω^{op}-like and centered.

Let $\mathcal{R}$ denote the set of nonempty regular open subsets of X. Choose $\langle W_n\rangle_{n<\omega} \in \mathcal{R}^{\omega}$ such that $\overline{W}_{n+1} \subsetneq W_n \neq X$ for all $n < \omega$. Let Ω denote the class of transfinite sequences $\langle\langle U_\alpha, V_\alpha\rangle\rangle_{\alpha<\eta}$ of elements of $\mathcal{R}^2$ satisfying the following.

(1) $\eta \geq \omega$ and $\langle\langle U_n, V_n\rangle\rangle_{n<\omega} = \langle\langle W_{n+1}, W_n\rangle\rangle_{n<\omega}$.
(2) $\overline{U}_\alpha \subseteq V_\alpha$ for all $\alpha < \eta$.
(3) $\mathcal{P}(V_\alpha) \cap \left\{\bigcap\sigma \setminus \overline{\bigcup\tau} : \sigma, \tau \in \left[\bigcup_{\beta<\alpha}\{U_\beta, V_\beta\}\right]^{<\omega}\right\} \subseteq \{\emptyset\}$ for all $\alpha < \eta$.
(4) $\Phi\left(\bigcup_{\alpha<\eta}\{U_\alpha, V_\alpha\}\right) = \emptyset$.

Seeking a contradiction, suppose η is a limit ordinal and $\langle\langle U_\alpha, V_\alpha\rangle\rangle_{\alpha<\eta} \notin \Omega$, but $\langle\langle U_\beta, V_\beta\rangle\rangle_{\beta<\alpha} \in \Omega$ for all $\alpha < \eta$. Then (1), (2), and (3) hold for $\langle\langle U_\alpha, V_\alpha\rangle\rangle_{\alpha<\eta}$, so there exists $\langle\sigma,\Gamma\rangle \in \Phi\left(\bigcup_{\alpha<\eta}\{U_\alpha, V_\alpha\}\right)$. We may choose $i \in \operatorname{dom}\Gamma$ such that $\Gamma(i) \not\subseteq \bigcup_{\beta<\alpha}\{U_\beta, V_\beta\}$ for all $\alpha < \eta$. Set $\Lambda = \Gamma \upharpoonright (\operatorname{dom}\Gamma \setminus \{i\})$. We may assume $\operatorname{dom}\Gamma$ is minimal among its possible values; hence, there exists $\tau \in \prod\Lambda$ such that $\bigcap\sigma \not\subseteq \overline{\bigcup\operatorname{ran}\tau}$. Choose $\alpha < \eta$ and $W \in \Gamma(i)$ such that $\sigma \cup \operatorname{ran}\tau \subseteq \bigcup_{\beta<\alpha}\{U_\beta, V_\beta\}$ and $W \in \{U_\alpha, V_\alpha\}$. Then $\bigcap\sigma \setminus \overline{\bigcup\operatorname{ran}\tau} \not\subseteq W$ by (2) and (3). Since W is regular, $\bigcap\sigma \setminus \overline{\bigcup\operatorname{ran}\tau} \not\subseteq \overline{W}$; hence, $\bigcap\sigma \not\subseteq \overline{W \cup \bigcup\operatorname{ran}\tau}$, in contradiction with $\langle\sigma,\Gamma\rangle \in \Phi\left(\bigcup_{\alpha<\eta}\{U_\alpha, V_\alpha\}\right)$. Thus, Ω is closed with respect to unions of increasing chains.

It follows from (3) that $\Omega \subseteq (\mathcal{R}^2)^{<|\mathcal{R}|^+}$. Moreover, $\langle\langle W_{n+1}, W_n\rangle\rangle_{n<\omega} \in \Omega$. Hence, by Zorn's Lemma, Ω has a maximal element $\langle\langle U_\alpha, V_\alpha\rangle\rangle_{\alpha<\eta}$. Set $\mathcal{B} = \bigcup_{\alpha<\eta}\{U_\alpha, V_\alpha\}$. Let us show that $\eta \geq \kappa$. Suppose not. For each $x \in X$, choose $Y_x, Z_x \in \mathcal{R}$ such that $x \in Y_x \subseteq \overline{Y}_x \subseteq Z_x$ and Z_x does not contain any nonempty open set of the form $\bigcap\sigma \setminus \overline{\bigcup\tau}$ where $\sigma, \tau \in [\mathcal{B}]^{<\omega}$. Choose $\rho \in [X]^{<\omega}$ such that $\bigcup_{x\in\rho} Y_x = X$. Let us show that $\Phi(\mathcal{B} \cup \{Y_x, Z_x\}) = \emptyset$ for some $x \in \rho$. Seeking a contradiction, suppose $\langle\sigma_x, \Gamma_x\rangle \in \Phi(\mathcal{B} \cup \{Y_x, Z_x\})$ for all $x \in \rho$. We may assume $\bigcup_{x\in\rho}\bigcup\operatorname{ran}\Gamma_x \subseteq \mathcal{B}$. Let Λ be a concatenation of $\{\Gamma_x : x \in \rho\}$ and set $\tau = \mathcal{B} \cap \bigcup_{x\in\rho}\sigma_i$. Then for all $\zeta \in \prod\Lambda$ we have

$$\bigcap\tau = \bigcap_{y\in\rho}\bigcap(\sigma_y \cap \mathcal{B}) = \bigcup_{x\in\rho}\left(Y_x \cap \bigcap_{y\in\rho}\bigcap(\sigma_y \cap \mathcal{B})\right) \subseteq \bigcup_{x\in\rho}\bigcap\sigma_x \subseteq \overline{\bigcup\operatorname{ran}\zeta}.$$

Hence, $\langle\tau,\Lambda\rangle \in \Phi(\mathcal{B})$, in contradiction with (4). Therefore, we may choose $x \in \rho$ such that $\Phi(\mathcal{B} \cup \{Y_x, Z_x\}) = \emptyset$. But then $\langle\langle U_\alpha, V_\alpha\rangle\rangle_{\alpha<\eta+1} \in \Omega$ if we set $U_\eta = Y_x$ and $V_\eta = Z_x$, in contradiction with the maximality of $\langle\langle U_\alpha, V_\alpha\rangle\rangle_{\alpha<\eta}$. Thus, $\eta \geq \kappa$.

Set $\mathcal{A} = \{V_\alpha : \alpha < \eta\}$. By (3), $|\mathcal{A}| = |\eta| \geq \kappa$. Set $K = \bigcap_{\alpha<\eta}\overline{U}_\alpha$. Then it suffices to show that $\mathcal{A}$ is a local $\langle|\eta|,\omega\rangle$-splitter at some $x \in K$. Suppose not. Then each $x \in K$ has an open neighborhood W_x that is a subset of infinitely many elements of $\mathcal{A}$. Hence, $\Phi(\mathcal{B} \cup \{W_x\}) \neq \emptyset$ for all $x \in K$. Choose $\rho \in [K]^{<\omega}$ such that $K \subseteq \bigcup_{x\in\rho} W_x$. Choose an open set W such that $W \cup \bigcup_{x\in\rho} W_x = X$ and $\overline{W} \cap K = \emptyset$. By compactness, $\mathcal{B} \cup \{W\}$ is not centered; hence, $\Phi(\mathcal{B} \cup \{W\}) \neq \emptyset$. Reusing our earlier concatenation argument, we have $\Phi(\mathcal{B}) \neq \emptyset$, in contradiction with (4). Thus, $\mathcal{A}$ is a local $\langle|\eta|,\omega\rangle$-splitter at some $x \in K$. $\square$

Lemma 5.3. *Suppose E is a subset of a space X and E has no finite neighborhood base. Then $\chi Nt(E, X)$ is the least $\kappa \geq \omega$ for which there is a local $\langle \chi(p, X), \kappa \rangle$-splitter at E.*

Proof. Set $\kappa = \chi Nt(E, X)$ and $\lambda = \chi(E, X)$. By Lemma 2.4, $\lambda \geq \kappa$; hence, a κ^{op}-like neighborhood base of E (which necessarily has size λ) is a local $\langle \lambda, \kappa \rangle$-splitter at E. To show the converse, let $\langle U_\alpha \rangle_{\alpha < \lambda}$ be a sequence of open neighborhoods of E. Let $\{V_\alpha : \alpha < \lambda\}$ be a neighborhood base of E. For each $\alpha < \lambda$, choose $W_\alpha \in \{V_\beta : \beta < \lambda\}$ such that $W_\alpha \subseteq U_\alpha \cap V_\alpha$. Then $\{W_\alpha : \alpha < \lambda\}$ is a neighborhood base of E. Let $\mu < \kappa$. Then there exist $\alpha < \lambda$ and $I \in [\lambda]^\mu$ such that $W_\alpha \subseteq \bigcap_{\beta \in I} W_\beta$. Hence, E is contained in the interior of $\bigcap_{\beta \in I} U_\beta$. Hence, $\{U_\alpha : \alpha < \lambda\}$ is not a local $\langle \lambda, \mu \rangle$-splitter at E. $\qquad\square$

Proof of Theorem 2.14. We may assume $\chi(X) \geq \omega_1$. By Theorem 5.2, there is a local $\langle \chi(X), \omega \rangle$-splitter at some $p \in X$. By Lemma 5.3, $\chi Nt(p, X) = \omega$. $\qquad\square$

Proof of Theorem 1.7. Let X be a homogeneous compactum. By a result of Arhangel'skiĭ (see 1.5 of [1]), $|Y| \leq 2^{\pi\chi(Y)c(Y)}$ for all homogeneous spaces Y. Since $|X| = 2^{\chi(X)}$ by Arhangel'skiĭ's Theorem and the Čech-Pospišil Theorem, we have $\chi(X) \leq \pi\chi(X)c(X)$ by GCH. If $\pi\chi(X) = \chi(X)$, then $\chi Nt(X) = \omega$ by Theorem 2.14. Hence, we may assume $\pi\chi(X) < \chi(X)$; hence, $\chi Nt(X) \leq \chi(X) \leq c(X)$ by Theorem 2.5. $\qquad\square$

Example 5.4. Consider 2^{ω_1} ordered lexicographically. Every point in this space has character and local Noetherian type ω_1, and some but not all points have π-character ω.

Definition 5.5 (Tukey [29]). Given two quasiorders P and Q, we say f is a *Tukey map* from P to Q and write $f \colon P \leq_T Q$ if f is a map from P to Q such that all preimages of bounded subsets of Q are bounded in P. We say that P is *Tukey reducible* to Q and write $P \leq_T Q$ if there exists $f \colon P \leq_T Q$. We say that P and Q are *Tukey equivalent* and write $P \equiv_T Q$ if $P \leq_T Q \leq_T P$.

Tukey showed that two directed sets are Tukey equivalent if and only if they embed as cofinal subsets of a common directed set. In particular, any two local bases at a common point in a topological space are Tukey equivalent. Another, easily checked fact is thats $P \leq_T [\operatorname{cf} P]^{<\omega}$ for every directed set P. Also, $[\kappa]^{<\omega} \leq_T [\lambda]^{<\omega}$ if $\kappa \leq \lambda$.

Lemma 5.6. *Suppose $\kappa \geq \omega$ and E is a subset of a space X with a local $\langle \kappa, \omega \rangle$-splitter at E. Then $\langle [\kappa]^{<\omega}, \subseteq \rangle \leq_T \langle \mathcal{A}, \supseteq \rangle$ for every neighborhood base $\mathcal{A}$ of E.*

Proof. Let $\mathcal{U}$ be a local $\langle \kappa, \omega \rangle$-splitter at E. Let $\mathcal{N}$ be the set of open neighborhoods of E. Then $\mathcal{N}$ is Tukey equivalent to every neighborhood base of E (with respect to $\supseteq$), so it suffices to show that $[\mathcal{U}]^{<\omega} \leq_T \langle \mathcal{N}, \supseteq \rangle$. Define $f \colon [\mathcal{U}]^{<\omega} \to \mathcal{N}$ by $f(\sigma) = \bigcap \sigma$ for all $\sigma \in [\mathcal{U}]^{<\omega}$. Then, for all $N \in \mathcal{N}$, we have $|f^{-1}{\uparrow}N| < \omega$ because $\mathcal{U}$ is a local $\langle \kappa, \omega \rangle$-splitter; whence, $f^{-1}{\uparrow}N$ is bounded in $[\mathcal{U}]^{<\omega}$. Thus, $f \colon [\mathcal{U}]^{<\omega} \leq_T \langle \mathcal{N}, \supseteq \rangle$. $\qquad\square$

Theorem 5.7. *Suppose X is a compactum and $\omega_1 \leq \kappa = \min_{p \in X} \pi\chi(p, X)$. Then, for some $p \in X$, every local base $\mathcal{A}$ at p satisfies $\langle [\kappa]^{<\omega}, \subseteq \rangle \leq_T \langle \mathcal{A}, \supseteq \rangle$.*

Proof. Combine Theorem 5.2 and Lemma 5.6. $\qquad\square$

Lemma 5.8. *Suppose E is a subset of a space X and E has no finite neighborhood base. Then the following are equivalent.*

(1) *$\chi Nt(E, X) = \omega$.*
(2) *There is a local $\langle \chi(E,X), \omega \rangle$-splitter at E.*
(3) *Every neighborhood base $\mathcal{A}$ of E satisfies $\langle [\chi(E,X)]^{<\omega}, \subseteq \rangle \equiv_T \langle \mathcal{A}, \supseteq \rangle$.*

Proof. By Lemma 5.3, (1) and (2) are equivalent. Let $\mathcal{B}$ be a neighborhood base of E of size $\chi(E, X)$. By Lemma 5.6, (2) implies $[\chi(E,X)]^{<\omega} \leq_T \langle \mathcal{A}, \supseteq \rangle \equiv_T \langle \mathcal{B}, \supseteq \rangle \leq_T [\chi(E,X)]^{<\omega}$ for every neighborhood base $\mathcal{A}$ of E. Thus, (2) implies (3). Finally, suppose $\mathcal{A}$ is a neighborhood base of E and $[\chi(E,X)]^{<\omega} \equiv_T \langle \mathcal{A}, \supseteq \rangle$. Then $[\chi(E,X)]^{<\omega}$ and $\langle \mathcal{A}, \supseteq \rangle$ embed as cofinal subsets of a common directed set. Hence, $\langle \mathcal{A}, \subseteq \rangle$ is almost ω^{op}-like by Lemma 2.21. Hence, $\mathcal{A}$ contains an ω^{op}-like neighborhood base of E. Thus, (3) implies (1). $\square$

Theorem 5.9. *Suppose X is an infinite homogeneous compactum and $\pi\chi(X) = \chi(X)$. Then, for all $p \in X$ and for all local bases $\mathcal{A}$ at p, we have $\langle \mathcal{A}, \supseteq \rangle \equiv_T \langle [\chi(X)]^{<\omega}, \subseteq \rangle$.*

Proof. Combine Theorem 2.14 and Lemma 5.8. $\square$

Definition 5.10. Given $n < \omega$ and ordinals $\alpha, \beta_0, \ldots, \beta_n$, let $\alpha \to (\beta_0, \ldots, \beta_n)$ denote the proposition that for all $f \colon [\alpha]^2 \to n+1$ there exist $i \leq n$ and $H \subseteq \alpha$ such that $f[[H]^2] = \{i\}$ and H has order type β_i.

Lemma 5.11. *Suppose $\kappa = \operatorname{cf} \kappa > \omega$ and P is a directed set such that $[\kappa]^{<\omega} \leq_T P$. Then P contains a set of κ-many pairwise incomparable elements.*

Proof. Let Q be a well-founded, cofinal subset of P. Then $P \equiv_T Q$; let $f \colon [\kappa]^{<\omega} \leq_T Q$. Define $g \colon [\kappa]^2 \to 3$ by $g(\{\alpha < \beta\}) = 0$ if $f(\{\alpha\}) \not\leq f(\{\beta\}) \not\leq f(\{\alpha\})$ and $g(\{\alpha < \beta\}) = 1$ if $f(\{\alpha\}) > f(\{\beta\})$ and $g(\{\alpha < \beta\}) = 2$ if $f(\{\alpha\}) \leq f(\{\beta\})$. By the Erdös-Dushnik-Miller Theorem, $\kappa \to (\kappa, \omega+1, \omega+1)$. Since Q is well-founded, there is no $H \in [\kappa]^\omega$ such that $g[[H]^2] = \{1\}$. Since f is Tukey and all infinite subsets of $[\kappa]^{<\omega}$ are unbounded, there is no $H \subseteq \kappa$ of order type $\omega+1$ such that $g[[H]^2] = \{2\}$. Hence, there exists $H \in [\kappa]^\kappa$ such that $g[[H]^2] = \{0\}$; whence, $f[[H]^1]$ is a κ-sized, pairwise incomparable subset of P. $\square$

Theorem 5.12. *Suppose $\kappa = \operatorname{cf} \kappa > \omega$ and X is a compactum such that every point has a local base not containing a set of κ-many pairwise incomparable elements. Then some point in X has π-character less than κ.*

Proof. Combine Theorem 5.7 and Lemma 5.11 to prove the contrapositive of the theorem. $\square$

Corollary 5.13. *Suppose X is a compactum such that every point has a local base that is well quasi-ordered with respect to $\supseteq$. Then some point in X has countable π-character.*

Finally, let us present a few results about local Noetherian type and topological embeddings.

Lemma 5.14. *Suppose X is a space, $Y \subseteq X$, and $p \in Y$ satisfies $\chi(p, Y) = \chi(p, X)$. Then $\chi Nt(p, X) \leq \chi Nt(p, Y)$.*

Proof. Set $\lambda = \chi(p, Y)$ and $\kappa = \chi Nt(p, Y)$; we may assume $\lambda > \omega$ by Theorem 2.5. By Lemma 5.3, we may choose a local $\langle \lambda, \kappa \rangle$-splitter $\mathcal{A}$ at p in Y. For each $U \in \mathcal{A}$, choose an open subset $f(U)$ of X such that $f(U) \cap Y = U$. Set $\mathcal{B} = f[\mathcal{A}]$. Then $|\mathcal{B}| = \lambda$ because f is bijective. Suppose $\mathcal{C} \in [\mathcal{B}]^\kappa$ and p is in the interior of $\bigcap \mathcal{C}$ with respect to X. Then p is in the interior of $Y \cap \bigcap \mathcal{C}$ with respect to Y, in contradiction how we chose $\mathcal{A}$. Thus, $\mathcal{B}$ is a local $\langle \lambda, \kappa \rangle$-splitter at p in X. By Lemma 5.3, $\chi Nt(p, X) \leq \kappa$. $\square$

Definition 5.15. For all infinite cardinals κ, let $u(\kappa)$ denote the space of uniform ultrafilters on κ.

Theorem 5.16. *For each $\kappa \geq \omega$, there exists $p \in u(\kappa)$ such that $\chi Nt(p, u(\kappa)) = \omega$ and $\chi(p, u(\kappa)) = 2^\kappa$.*

Proof. Generalizing an argument of Isbell [12] about $\beta\omega$, let A be an independent family of subsets of κ of size 2^κ. Set $B = \bigcup_{F \in [A]^\omega} \{x \subseteq \kappa : \forall y \in F \ |x \backslash y| < \kappa\}$. Since A is independent, we may extend A to an ultrafilter p on κ such that $p \cap B = \emptyset$. For each $x \subseteq \kappa$, set $x^* = \{q \in u(\kappa) : x \in q\}$. Then $\{x^* : x \in A\}$ is a local $\langle 2^\kappa, \omega \rangle$-splitter at p. Since $\chi(p, u(\kappa)) \leq 2^\kappa$, it follows from Lemma 5.3 that $\chi Nt(p, u(\kappa)) = \omega$ and $\chi(p, u(\kappa)) = 2^\kappa$. $\square$

Theorem 5.17. *Suppose $\kappa \geq \omega$ and X is a space such that $\chi(X) = 2^\kappa$ and $u(\kappa)$ embeds in X. Then there is an ω^{op}-like local base at some point in X. Hence, $\chi Nt(X) = \omega$ if X is homogeneous.*

Proof. Let j embed $u(\kappa)$ into X. By Theorem 5.16, there exists $p \in u(\kappa)$ such that $\chi Nt(p, u(\kappa)) = \omega$ and $\chi(p, u(\kappa)) = 2^\kappa$. By Lemma 5.14, $\chi Nt(j(p), X) = \omega$. $\square$

Theorem 5.18. *Suppose p is a point in a dense subspace Y of a T_3 space X. Then $\chi Nt(p, X) \geq \chi Nt(p, Y)$.*

Proof. Set $\kappa = \chi Nt(p, Y)$ and let $\mathcal{A}$ be a κ^{op}-like local base at p in X. By Lemma 2.21, we may assume $\mathcal{A}$ consists only of regular open sets. Set $\mathcal{B} = \{U \cap Y : U \in \mathcal{A}\}$. Given any $U, V \in \mathcal{A}$ such that $U \not\subseteq V$, we have $U \backslash \overline{V} \neq \emptyset$; whence, $U \cap Y \backslash \overline{V} \neq \emptyset$; whence, $U \cap Y \not\subseteq V \cap Y$. Therefore, $\mathcal{B}$ is κ^{op}-like; hence, $\chi Nt(p, Y) \leq \chi Nt(p, X)$. $\square$

Example 5.19. Consider the sequential fan Y with ω-many spines. More explicitly, Y is the space $\omega^2 \cup \{p\}$ obtained by taking $\omega \times (\omega + 1)$ and collapsing the subspace $\omega \times \{\omega\}$ to a point p. It is easily checked that Y is $T_{3.5}$. Choose a compactification X of Y. Then $c(X) = c(Y) = \omega$ and X is not homogeneous because it has isolated points. We will show $\chi Nt(p, X) \geq \omega_1$, thereby demonstrating that homogeneity cannot be removed from the hypothesis of Theorem 1.7. It suffices to show that $\chi Nt(p, Y) \geq \omega_1$, for we can then apply Theorem 5.18. Given $f \in \omega^\omega$, set $U_f = \{p\} \cup \{\langle m, n \rangle \in \omega^2 : n \geq f(m)\}$. Set $\mathcal{A} = \{U_f : f \in \omega^\omega\}$, which is a local base at p in Y. Suppose $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{B}$ is a local base at p. Then it suffices to show that $\mathcal{B}$ is not ω^{op}-like. By an easy diagonalization argument, no local base at p is countable. Choose $\mathcal{B}_0 \in [\mathcal{A}]^{\omega_1}$. Given $n < \omega$, $\mathcal{B}_n \in [\mathcal{B}]^{\omega_1}$, and $U_{f_0}, \ldots, U_{f_{n-1}} \in \mathcal{B}$, choose $\mathcal{B}_{n+1} \in [\mathcal{B}_n]^{\omega_1}$ such that $g(n) = h(n)$ for all $U_g, U_h \in \mathcal{B}_{n+1}$. Then choose $U_{f_n} \in \mathcal{B}_{n+1} \backslash \{U_{f_0}, \ldots, U_{f_{n-1}}\}$. For each $n < \omega$, set $g(n) = \max\{f_0(n), \ldots, f_n(n)\}$ Then $U_g \subseteq U_{f_n}$ for all $n < \omega$; hence, $\mathcal{B}$ is not ω^{op}-like.

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