ON THE STRONG FREESE-NATION PROPERTY

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Abstract. We show that there is a boolean algebra that has the Freese-Nation property (FN) but not the strong Freese-Nation property (SFN), thus answering a question of Heindorf and Shapiro. Along the way, we produce some new characterizations of the FN and SFN in terms of sequences of elementary submodels.

1. The Freese-Nation property and friends

Definition 1.1.

- Given a poset P and a map f from P to the power set of P, we say that f is *interpolating* if, for all pairs $x \leq_P y$, there exists $z \in [x, y] \cap f(x) \cap f(y)$.
- We say that poset P has the *Freese-Nation property (FN)* if there is an interpolating map from P to $[P]^{<\aleph_0}$, the set of finite subsets of P. Such a map is called an *FN map.*

When a P is also a boolean algebra, the FN can be understood as an abstraction of the Interpolation Theorem of propositional logic, which states that if the implication $\varphi \to \psi$ is tautological for two propositions φ and ψ, then there is a proposition χ such that $\varphi \to \chi$ and $\chi \to \psi$ are tautological and the propositional variables of χ are common to φ and ψ. An easy consequence of the Interpolation Theorem is that free boolean algebras have the FN.

The FN is named after Freese and Nation [2], who introduced it in 1978 as part of a characterization of projective lattices. In particular, every projective lattice has the FN (but the converse was already known to be false, even for finite lattices). Since the morphisms in the category of lattices and lattice homomorphisms are epimorphisms if and only if they are surjective, a lattice is projective if and only if it is a retract of a free lattice. Likewise, a boolean algebra is projective if and only if it is a retract of a free boolean algebra. The Stone duals of the projective boolean algebras are exactly the Dugundji spaces, *i.e.*, the continuous retracts of powers of 2.

The Stone duals of the boolean algebras with the FN were elegantly characterized in two ways by Ščepin [10, 11], as the existence of a distance function between points and regular closed sets and as the existence of a rich family of open quotient maps. Succinctly, a compact Hausdorff space is "k-metrizable" if and only if it is "openly generated;" a boolean space is openly generated if and only if its clopen algebra has the FN. Ščepin also proved that every Dugundji space is openly generated, that

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the Vietoris hyperspace operation preserves open generation, and that every openly generated boolean space of weight at most $\aleph_1$ is Dugundji. However, Shapiro [9] proved that the Vietoris hyperspace of 2^κ is not a continuous image of a power of 2 if $\kappa \geq \aleph_2$. Thus, for boolean algebras up to size $\aleph_1$, the FN is equivalent to projectivity, while for boolean algebras in general, projectively strictly implies the FN.

Fuchino translated Ščepin's notion of openly generated into the language of elementary substructures in an appendix to [4]. Before we can state this characterization, we need a few definitions.

Definition 1.2.

- If P is a poset, S is a set, and $p \in P$, then, when they exist, let
  - $\pi_+^S(p) = \min\{q \in P \cap S : q \geq p\}$ and
  - $\pi_-^S(p) = \max\{q \in P \cap S : q \leq p\}$.
- Given a poset P and $Q \subseteq P$, we say that Q is a *relatively complete* suborder of P if, for all $p \in P$, $\pi_+^Q(p), \pi_-^Q(p)$ exist.
- Given boolean algebras A and B, we write $A \leq B$ to indicate that A is a subalgebra of B.
- If B is a boolean algebra, $A \leq B$, and A is a relatively complete suborder of B, then we write $A \leq_{\mathrm{rc}} B$.
- If $\psi\colon A \to B$ is a boolean homomorphism, we say that ψ is relatively complete if $\psi[A] \leq_{\mathrm{rc}} B$.

Note that the topological dual of a relatively complete boolean homomorphism is an open map between two boolean spaces.

Definition 1.3.

- Given two sets P and Q, we write $P \prec Q$ if $(P, \in)$ is an elementary substructure of $(Q, \in)$.
- Given a cardinal μ, let $H(\mu)$ denote the set of all sets with transitive closure smaller than μ.

Given a boolean algebra $\mathcal{A} = (A, 0, 1, \wedge, \vee, -)$, we will abuse notation by using A to denote both A and $\mathcal{A}$. In particular, when we write $A \in M$ for some set M, we mean $\mathcal{A} \in M$.

Theorem 1.1 (Fuchino). *Let A be a boolean algebra and let μ be a regular uncountable cardinal such that $A \in H(\mu)$. The following are then equivalent.*

(1) *A has the FN.*
(2) *$A \cap M \leq_{\mathrm{rc}} A$ for all countable M satisfying $A \in M \prec H(\mu)$.*
(3) *$A \cap M \leq_{\mathrm{rc}} A$ for all M satisfying $A \in M \prec H(\mu)$.*

If we weaken the definition of FN map to allow as outputs countable sets instead of merely finite sets, then we obtain the weak Freese-Nation property (WFN), which was initially investigated in topological terms by Ščepin [11] and later systematically studied in Heindorf and Shapiro's 1994 book *Nearly Projective Boolean Algebras* [4]. For our purposes, their most interesting result about the WFN is a characterization of it as the existence of a rich family of commuting subalgebras. Elementary substructure characterizations analogous to (2) and (3) from the previous theorem were proved by Fuchino, Koppelberg, and Shelah in [3] and by the author in [8], respectively.

Definition 1.4.

- Given a poset P and $A, B \subseteq P$, we say that A and B *commute*, writing $A \downarrow B$, if, for all pairs $(x, y) \in A \times B$, if $x \leq y$, then $[x, y] \cap A \cap B$ is nonempty, and if $y \leq x$, then $[y, x] \cap A \cap B$ is nonempty.
- Given a poset P and $Q \subseteq P$, we say that $Q \subseteq_\sigma P$ if, for all $p \in P$, there exist countable sets $L(p), U(p) \subseteq Q$ such that
  - $\{q \in Q : q \leq p\} = \bigcup_{r \in L(p)} \{q \in Q : q \leq r\}$ and
  - $\{q \in Q : q \geq p\} = \bigcup_{r \in U(p)} \{q \in Q : q \geq r\}$.

Note that if A and B are subalgebras of a boolean algebra C, then $A \downarrow B$ if and only if, for all ultrafilters U of A and V of B, if $U \cap B = V \cap A$, then $U \cup V$ extends to an ultrafilter of C.

Theorem 1.2. *Let A be a boolean algebra and let μ be a regular uncountable cardinal such that $A \in H(\mu)$. The following are then equivalent.*

1. *A has the WFN, i.e., there is an interpolating map from A to $[A]^{<\aleph_1}$.*
2. *(Fuchino, Koppelberg, Shelah) $A \cap M \subseteq_\sigma A$ for all M satisfying $A \in M \prec H(\mu)$ and $|M| = \omega_1 \subseteq M$.*
3. *$A \cap M \subseteq_\sigma A$ for all M satisfying $A \in M \prec H(\mu)$ and $\omega_1 \subseteq M$.*
4. *(Ščepin, Heindorf, Shapiro) There is a cofinal family $\mathcal{C}$ of countable subalgebras of A such that $F \downarrow G$ for all $F, G \in \mathcal{C}$.*

In [4], Heindorf and Shapiro defined the natural analog of (4) for the FN to be the strong Freese-Nation property (SFN).

Definition 1.5. A boolean algebra has the *SFN* if and only if it has a pairwise commuting cofinal family of finite subalgebras.

Also in [4], Heindorf and Shapiro showed that projectivity implies the SFN implies the FN. Hence, the three properties are equivalent for boolean algebras of size at most $\aleph_1$. They further showed that the symmetric square and exponential operations preserve the SFN; Ščepin had already shown the same for the FN [10]. On the other hand, if $\kappa \geq \aleph_2$, then the exponential [9] and symmetric square [11] of a free boolean algebra of size κ are not projective. Thus, the two most natural examples of non-projective boolean algebras with the FN also have the SFN. Naturally, Heindorf and Shapiro posed the question of whether the SFN is actually equivalent to the FN. Twenty years later, there appears to have been no subsequent published work on the SFN. The primary motivation of this work is to answer Heindorf and Shapiro's question.

Theorem 1.3. *There is a boolean algebra of size $\aleph_2$ that has the FN but not the SFN.*

To prove the above, we require new characterizations of the FN and SFN in terms of "retrospective" sequences of countable elementary submodels, as we shall explain shortly. We expect that the techniques we use here for separating the FN and SFN will see wider application in the future, and have stated many lemmas in anticipating generality.

For additional background information about the classes of boolean algebras defined by the SFN, FN, WFN, and projectivity, we refer the reader to [4].

2. Retrospective sequences of elementary substructures

Our proof of Theorem 1.3 uses long λ-approximation sequences, which one can think of as a poor man's higher-gap morasses, available in ZFC. These sequences were introduced in [7] as a more flexible version of Davies' trees of substructures [1]. Davies used such a tree to prove that the plane is a union of countably many rotated graphs of functions; Jackson and Mauldin [5] used such a tree to prove that there exists a subset of the plane intersecting every isometric copy of $\mathbb{Z}^2$ at exactly one point. The main application of long λ-approximation sequences in [7] was to prove that, for a class of topological spaces that includes every compact group, every topological base of a space contains a base of the same space which is upper finite with respect to inclusion.

Definition 2.1.

- Call a sequence of sets $(A_i)_{i\in I}$ *retrospective* if I is an ordinal and, for all $i \in I$, the sequence $(A_j)_{j<i}$ is an element of A_i.
- Given μ a regular uncountable cardinal and λ a regular uncountable cardinal at most μ, call a set M *λ-approximating* if $|M| < \lambda$, $M \cap \lambda \in \lambda$, and $M \prec H(\mu)$.
- Given μ and λ as above, and η an ordinal at most μ, a transfinite sequence $(M_i)_{i<\eta}$ is called a *long λ-approximation sequence* if it is retrospective and M_i is λ-approximating for all $i < \eta$.

(In Definition 3.16 of [7], it was required of long λ-approximation sequences that also $|M_i| \subseteq M_i$ and $\lambda \in M_i$. Here, we do not require $|M_i| \subseteq M_i$ because it is not needed for any applications (so far). We do not require $\lambda \in M_i$ for the same reason, and because if $\lambda \leq i < \eta$, then λ is definable in M_i as $\sup_{j<i} \min(i \setminus M_j)$.)

The requirement $M_i \cap \lambda \in \lambda$ is succinct but perhaps obscures its intended application, which is that for all $A \in M_i$, if $|A| < \lambda$, then $A \subseteq M_i$. In particular, if $i < \lambda$, then $\bigcup_{j<i} M_j \subseteq M_i$. Also notice that the requirement $M_i \cap \lambda \in \lambda$ is redundant if $\lambda = \omega_1$.

Lemma 2.1. *Given regular uncountable cardinals $\lambda \leq \mu$, $A \in [H(\mu)]^{<\lambda}$, $\eta < \mu$, and $(M_i)_{i<\eta}$ a long λ-approximation sequence, there exists M_η such that $A \subseteq M_\eta$ and $(M_i)_{i<\eta+1}$ is a long λ-approximation sequence.*

Proof. Let $B = A \cup \{(M_i)_{i<\eta}\}$ and choose $M_\eta = \bigcup_{n<\omega} N_n$ where $B \subseteq N_0$, $|N_n| < \lambda$, $N_n \prec H(\mu)$, and $N_n \cup \sup(\lambda \cap N_n) \subseteq N_{n+1}$ for all n. □

Lemma 2.2. *Given $(M_i)_{i<\eta}$ as in the above definition and $\alpha, \beta < \eta$, the following are equivalent.*

(1) $\alpha \in \beta \cap M_\beta$
(2) $M_\alpha \in M_\beta$
(3) $M_\alpha \subsetneq M_\beta$

Proof. Given (1), we have (2) because M_α is definable from α and $(M_\gamma)_{\gamma<\beta}$. Given (2), we have $M_\alpha \subseteq M_\beta$ because $|M_\alpha| \in M_\beta \cap \lambda \in \lambda$; we also have $M_\alpha \in M_\beta \setminus M_\alpha$. Given (3), we have $\alpha \neq \beta$; we also have $\alpha \in M_\beta$ because α is definable in M_α from $(M_\gamma)_{\gamma<\alpha}$; we also have $\alpha \leq \beta$ because otherwise $M_\alpha \subseteq \bigcup_{\gamma<\alpha} M_\gamma$, which is impossible because $H(\mu) \not\subseteq \bigcup_{\gamma<\alpha} M_\gamma$ and $\bigcup_{\gamma<\alpha} M_\gamma \in M_\alpha \prec H(\mu)$. □

Lemma 2.3. *If $S \in M_0$ and $(M_i)_{i<|S|}$ is a long λ-approximation sequence, then $S \subseteq \bigcup_{i<|S|} M_i$.*

Proof. Some $f \in M_0$ is a surjection from $|S|$ to S. By Lemma 2.2, $M_0 \subsetneq M_\alpha$ for all $\alpha > 0$, so $f(\alpha) \in M_\alpha$ for all $\alpha < |S|$. □

Definition 2.2. Given an ordinal α and an infinite cardinal λ, let the *λ-truncated cardinal normal form of* α denote the unique polynomial

$$\omega_{\beta_0}\gamma_0 + \cdots + \omega_{\beta_{m-1}}\gamma_{m-1} + \gamma_m$$

equal to α and satisfying $\omega_{\beta_0} > \cdots > \omega_{\beta_{m-1}} \geq \lambda$, $\gamma_i \in [1, \omega_{\beta_i}^+)$ for all $i < m$, and $\gamma_m < \lambda$. For each $i < m$, let $\partial_i\alpha$ denote $\omega_{\beta_i}\gamma_i$ and let $\lfloor\alpha\rfloor_{i+1}$ denote $\sum_{j<i+1}\partial_j\alpha$; let $\lfloor\alpha\rfloor_0 = 0$ and $\lfloor\alpha\rfloor_{m+1} = \alpha$. Let $\urcorner(\alpha)$ denote m if $\gamma_m = 0$; $m+1$ if $\gamma_m > 0$.

Observe that $\lfloor\alpha\rfloor_i$ is $\{\alpha\}$-definable in $H(|\alpha|^+)$ for each $i \leq \urcorner(\alpha)$. Hence, if $(M_\beta)_{\beta<\alpha}$ is a long λ-approximation sequence, $\zeta + \eta \leq \alpha$, and $\lfloor\zeta + \beta\rfloor_{\urcorner(\zeta)} = \zeta$ for all $\beta < \eta$, then, for each $\beta < \eta$, $(M_{\zeta+\gamma})_{\gamma<\beta}$ is definable in $M_{\zeta+\beta}$. Thus, such an $(M_{\zeta+\beta})_{\beta<\eta}$ is a long λ-approximation sequence. We will use this last observation to prove the fundamental lemma for long λ-approximation sequences, which is the existence of a definable finite partition into directed segments.

Lemma 2.4. *Given a long λ-approximation sequence $(M_\beta)_{\beta<\alpha}$, the sets $\{M_\beta : \lfloor\alpha\rfloor_i \leq \beta < \lfloor\alpha\rfloor_{i+1}\}$ are directed with respect to inclusion for all $i < \urcorner(\alpha)$.*

Proof. A proof is implicit in the proof of Lemma 3.17 of [7], but we will provide a shorter explicit proof here. Proceed by induction on α. If $\alpha \leq \lambda$, then $\{M_\beta : \beta < \alpha\}$ is a chain. If $\urcorner(\alpha) \geq 2$, then each $\{M_\beta : \lfloor\alpha\rfloor_i \leq \beta < \lfloor\alpha\rfloor_{i+1}\}$ is directed by our induction hypothesis applied to $(M_{\lfloor\alpha\rfloor_i+\beta})_{\beta<\partial_i\alpha}$. So, suppose that $\alpha > \lambda$ and $\urcorner(\alpha) = 1$. If $\alpha = \sup\{\beta < \alpha : \urcorner(\beta) = 1\}$, then $\{M_\gamma : \gamma < \alpha\}$ is directed because each $\{M_\gamma : \gamma < \beta\}$ is directed by induction. So, suppose that $\alpha = \kappa(\gamma + 1)$ where κ is a cardinal and $1 \leq \gamma < \kappa^+$. Set $\beta = \kappa\gamma$ and $S = \{M_\delta : \delta < \beta\}$. By Lemma 2.3 applied to S and $(M_{\beta+\delta})_{\delta<\kappa}$, we have $S \subset \bigcup_{\delta<\kappa} M_{\beta+\delta}$. Hence, by Lemma 2.2, for every $\varepsilon < \beta$ there exists $\delta < \kappa$ such that $M_\varepsilon \subsetneq M_{\beta+\delta}$. Therefore, $\{M_\delta : \delta < \alpha\}$ is directed because its cofinal subset $\{M_{\beta+\delta} : \delta < \kappa\}$ is directed by our inductive hypothesis applied to $(M_{\beta+\delta})_{\delta<\kappa}$. □

If $n < \omega$ and $(M_\alpha)_{\alpha<\lambda^{+n}}$ is a long λ-approximation sequence, then, since $\urcorner(\alpha) \leq n+1$ for all $\alpha < \lambda^{+n}$, we can sometimes use $(M_\alpha)_{\alpha<\lambda^{+n}}$ like a (λ, n)-morass, in the weak sense that we can build a λ^{+n}-sized object as a direct limit of small (that is, $(<\lambda)$-sized) pieces while locally only having to fit together at most $n+1$ small direct limits of these small pieces. Of course, we lack the additional coherence properties of a (λ, n)-morass, which require assumptions beyond ZFC. However, the citations given at the beginning of this section demonstrate that long λ-approximation sequence are useful even without such coherence. We will find them useful again in this paper. See also [13] for very recent additional applications, noting that there long ω_1-approximation sequences are called Davies sequences.

We finish this section with some additional lemmas about long λ-approximation sequences that we will need later.

Definition 2.3. Given a long λ-approximation sequence $(M_\beta)_{\beta<\eta}$, $\alpha \leq \eta$, and $i < \urcorner(\alpha)$, let

- $I_i(\alpha) = [\lfloor\alpha\rfloor_i, \lfloor\alpha\rfloor_{i+1})$;
- $I'_i(\alpha) = I_i(\alpha) \cap M_\alpha$;
- $\mathcal{I}_i(\alpha) = \{M_\beta : \beta \in I_i(\alpha)\}$;

- $\mathcal{I}'_i(\alpha) = \{M_\beta : \beta \in I'_i(\alpha)\}$;
- $M_{\alpha,i} = \bigcup \mathcal{I}_i(\alpha)$;
- $M'_{\alpha,i} = M_{\alpha,i} \cap M_\alpha$.

Lemma 2.5. *If $(M_\alpha)_{\alpha<\eta}$ is a long λ-approximation sequence, $i < \daleth(\eta)$, and $\partial_i \eta \geq \lambda$, then $|M_{\eta,i}| = \partial_i \eta \subseteq M_{\eta,i}$.*

Proof. Since $(M_{\lfloor\eta\rfloor_i+\alpha})_{\alpha<\partial_i\eta}$ is a long λ-approximation sequence, we have $\partial_i\eta \subseteq M_{\eta,i}$. Since each $M_{\lfloor\eta\rfloor_i+\alpha}$ is smaller than λ, we have $\partial_i\eta = |M_{\eta,i}|$. □

Lemma 2.6. *If $(M_\alpha)_{\alpha<\eta}$ is a long λ-approximation sequence and $S \in M_0$, then, for all $\alpha < \eta$, $S \in M_\alpha$ and $S \in M_{\alpha,i}$ for all $i < \daleth(\alpha)$.*

Proof. By Lemma 2.2, $M_0 \subseteq M_\alpha$ for all $\alpha < \eta$. Hence, also $M_0 \subseteq \bigcup_{\beta\in I_i(\alpha)} M_\beta$ for all $\alpha < \eta$ and $i < \daleth(\alpha)$. □

Definition 2.4. Because $\mathcal{I}_i(\alpha)$ and $\mathcal{I}'_i(\alpha)$ may not be downward closed in $\{M_\beta : \beta < \eta\}$ with respect to inclusion, we also define

- $J_i(\alpha) = \bigcup\{M_\beta \cap (\beta+1) : \beta \in I_i(\alpha)\}$;
- $J'_i(\alpha) = \bigcup\{M_\beta \cap (\beta+1) : \beta \in I'_i(\alpha)\}$;
- $\mathcal{J}_i(\alpha) = \{M_\beta : \beta \in J_i(\alpha)\}$;
- $\mathcal{J}'_i(\alpha) = \{M_\beta : \beta \in J'_i(\alpha)\}$.

By Lemma 2.2, $\mathcal{J}_i(\alpha)$ and $\mathcal{J}'_i(\alpha)$ are downward closed in $\{M_\beta : \beta < \eta\}$ with respect to inclusion. Also observe that, by elementarity and retrospectiveness,

- $\mathcal{I}'_i(\alpha) = \mathcal{I}_i(\alpha) \cap M_\alpha$;
- $J'_i(\alpha) = J_i(\alpha) \cap M_\alpha$;
- $\mathcal{J}'_i(\alpha) = \mathcal{J}_i(\alpha) \cap M_\alpha$.

Lemma 2.7. *Given a long λ-approximation sequence $(M_\beta)_{\beta<\alpha+1}$ and $i < \daleth(\alpha)$, $\mathcal{I}_i(\alpha)$ and $\mathcal{I}'_i(\alpha)$ are directed with respect to inclusion with respective unions $M_{\alpha,i}$ and $M'_{\alpha,i}$. Moreover, $\mathcal{I}_i(\alpha)$ is cofinal in $\mathcal{J}_i(\alpha)$ and $\mathcal{I}'_i(\alpha)$ is cofinal in $\mathcal{J}'_i(\alpha)$.*

Proof. By Lemma 2.4, $\mathcal{I}_i(\alpha)$ is directed; by definition, its union is $M_{\alpha,i}$. Since $(M_\beta : \beta \in I_i(\alpha)) \in M_\alpha \prec H(\mu)$, the set $\mathcal{I}_i(\alpha) \cap M_\alpha$ is directed with union $M_{\alpha,i} \cap M_\alpha$. Having thus proved the first sentence of the lemma, the second sentence immediately follows from Lemma 2.2. □

Lemma 2.8. *Given a long λ-approximation sequence $(M_\beta)_{\beta<\alpha+1}$, we have*

- $\bigcup_{\beta<\alpha} M_\beta = \bigcup_{i<\daleth(\alpha)} M_{\alpha,i}$,
- $M_\alpha \cap \bigcup_{\beta<\alpha} M_\beta = \bigcup_{i<\daleth(\alpha)} M'_{\alpha,i}$, *and*
- $M_{\alpha,i}, M'_{\alpha,i} \prec H(\mu)$ *for all* $i < \daleth(\alpha)$.

Proof. Clearly, $\alpha = \bigcup_{i<\daleth(\alpha)} I_i(\alpha)$; the two equations of the lemma immediately follow. By Lemmas 2.4 and 2.7, each $M_{\alpha,i}$ and each $M'_{\alpha,i}$ is a directed union of elementary substructures of $H(\mu)$, so $M_{\alpha,i}, M'_{\alpha,i} \prec H(\mu)$. □

Lemma 2.9. *Given a long λ-approximation sequence $(M_\beta)_{\beta<\alpha+1}$ and $i < \daleth(\alpha)$, we have $I'_i(\alpha) \not\subseteq \bigcup_{j\neq i} J_j(\alpha)$.*

Proof. Since λ is regular and $|I_j(\alpha)| > |I_k(\alpha)|$ for all $j < k < \daleth(\alpha)$, we have $|I_i(\alpha)| > \sum_{i<j<\daleth(\alpha)} |J_j(\alpha)|$. Let $\beta = \min\left(I_i(\alpha) \setminus \bigcup_{i<j<\daleth(\alpha)} J_j(\alpha)\right)$, which is definable in M_α and thus in $I'_i(\alpha) \setminus \bigcup_{i<j<\daleth(\alpha)} J_j(\alpha)$. Since $\beta \geq \lfloor\alpha\rfloor_i$, we also have $\beta \notin \bigcup_{j<i} J_j(\alpha)$. □

Definition 2.5. Given a long λ-approximation sequence $(M_\beta)_{\beta<\alpha+1}$ and nonempty $s \subseteq \daleth(\alpha)$, let

- $K_s(\alpha) = \bigcap_{i\in s} J_i(\alpha)$;
- $K'_s(\alpha) = \bigcap_{i\in s} J'_i(\alpha)$;
- $\mathcal{K}_s(\alpha) = \bigcap_{i\in s} \mathcal{J}_i(\alpha)$;
- $\mathcal{K}'_s(\alpha) = \bigcap_{i\in s} \mathcal{J}'_i(\alpha)$.

Observe that, by elementarity and retrospectiveness,

- $K'_s(\alpha) = K_s(\alpha) \cap M_\alpha$;
- $\mathcal{K}'_s(\alpha) = \mathcal{K}_s(\alpha) \cap M_\alpha$;
- $\mathcal{K}_s(\alpha) = \{M_\beta : \beta \in K_s(\alpha)\}$;
- $\mathcal{K}'_s(\alpha) = \{M_\beta : \beta \in K'_s(\alpha)\}$.

Lemma 2.10. *Given a long λ-approximation sequence $(M_\beta)_{\beta<\alpha+1}$ and nonempty $s \subseteq \daleth(\alpha)$, the sets $\mathcal{K}_s(\alpha)$ and $\mathcal{K}'_s(\alpha)$ are directed with respect to inclusion.*

Proof. Since $\mathcal{K}_s(\alpha) \in M_\alpha \prec H(\mu)$, it suffices to show that $\mathcal{K}_s(\alpha)$ is directed. Proceed by induction on $|s|$. Case $|s| = 1$ follows from Lemmas 2.4 and 2.7. Assuming $|s| > 1$, let $i = \max(s)$ and $t = s \setminus \{i\}$. Suppose that $\beta, \gamma \in K_s(\alpha)$. Since $\beta, \gamma \in J_i(\alpha)$, we have $M_\beta, M_\gamma \subseteq M_\delta$ for some $\delta \in I_i(\alpha)$. By definition, $K_t(\alpha) < I_i(\alpha)$; hence, $\beta, \gamma < \delta$; hence, $M_\beta, M_\gamma \in M_\delta$ by Lemma 2.2. Since $\delta \in I_i(\alpha)$, we have $I_j(\delta) = I_j(\alpha)$ for all $j < i$. Therefore, M_δ knows that $\mathcal{K}_t(\alpha)$ is directed and that $M_\beta, M_\gamma \in \mathcal{K}_t(\alpha)$. Hence, there exists $M_\varepsilon \in M_\delta \cap \mathcal{K}_t(\alpha)$ such that $M_\beta, M_\gamma \subseteq M_\varepsilon$. By Lemma 2.2, $\varepsilon \in M_\delta \cap \delta$; hence, $\varepsilon \in J_i(\alpha)$. Thus, M_β and M_γ have a common superset M_ε in $\mathcal{K}_t(\alpha) \cap \mathcal{J}_i(\alpha)$, as desired. $\square$

Definition 2.6. Given a long λ-approximation sequence $(M_\beta)_{\beta<\eta}$ and $x \in \bigcup_{\beta<\eta} M_\beta$, let the *$M$-rank of x*, written $\rho(x, M)$ or just $\rho(x)$, denote the least $\alpha < \eta$ such that $x \in M_\alpha$.

Lemma 2.11. *Given a long λ-approximation sequence $(M_\beta)_{\beta<\alpha+1}$ and $x \in M_\alpha$, we have $M_{\rho(x)} \subseteq M_\alpha$.*

Proof. Supposing $\rho(x) < \alpha$, we have $M_{\rho(x)}$ definable in M_α from x and $(M_\beta)_{\beta<\alpha}$. By Lemma 2.2, we then have $M_{\rho(x)} \subsetneq M_\alpha$. $\square$

Lemma 2.12. *For every long λ-approximation sequence $(M_\beta)_{\beta<\eta}$ and $\varnothing \neq E \subseteq \eta$, there exists $D \subseteq \eta$ such that $\bigcap_{\alpha\in E} M_\alpha = \bigcup_{\alpha\in D} M_\alpha$ and $\{M_\alpha : \alpha \in D\}$ is directed.*

Proof. Let $N = \bigcap_{\alpha\in E} M_\alpha$. By Lemma 2.11, $N = \bigcup\{M_\alpha : \alpha < \eta \text{ and } M_\alpha \subseteq N\}$. Suppose that $\alpha, \beta < \eta$ and $M_\alpha, M_\beta \subsetneq N$. It suffices to find $\gamma < \eta$ such that $M_\alpha \cup M_\beta \subseteq M_\gamma \subseteq N$. First, note that $M_\alpha, M_\beta \in M_i$ for all $i \in E$ by Lemma 2.2. Since E is nonempty, we may define $\gamma = \rho(\{M_\alpha, M_\beta\})$. We then have $M_\alpha, M_\beta \subsetneq M_\gamma$ (again by Lemma 2.2). Fix $i \in E$; it suffices to show that $M_\gamma \subseteq M_i$. By definition of ρ, $\gamma \leq i$. If $\gamma < i$, then $M_\gamma \subsetneq M_i$ because M_γ is definable in M_i. $\square$

Lemma 2.13. *For every long λ-approximation sequence $(M_\alpha)_{\alpha<\eta}$ and $E \subseteq \eta$, if $\{M_\alpha : \alpha \in E\}$ is directed, then there exists $i < \daleth(\eta)$ such that $E \subseteq J_i(\eta)$.*

Proof. Let $\mathcal{E} = \{M_\alpha : \alpha \in E\}$ and $\mathcal{E}_i = \{M_\alpha : \alpha \in E \cap I_i(\eta)\}$ for each $i < \daleth(\eta)$. Since $\mathcal{E}$ is directed and $\{\mathcal{E}_i : i < \daleth(\eta)\}$ is a finite partition of $\mathcal{E}$, there must exist i such that $\mathcal{E}_i$ is cofinal in $\mathcal{E}$. By Lemma 2.2, $E \subseteq J_i(\eta)$ for any such i. $\square$

3. Retrospective characterizations of the FN and SFN

Lemma 3.1. *Given a poset C and $A, B \subseteq C$ such that $A \downarrow B$ and $A \cap B$ is a relatively complete suborder of A, the functions $\pi_+^B \upharpoonright A$ and $\pi_-^B \upharpoonright A$ respectively equal $\pi_+^{A\cap B} \upharpoonright A$ and $\pi_-^{A\cap B} \upharpoonright A$.*

Proof. Given $a \in A$ and $b \in B$ such that $a \leq b$, we have some $c \in [a, b] \cap A \cap B$; hence, $\pi_+^{A\cap B}(a) \leq c \leq b$; hence, $\pi_+^B(a)$ exists and equals $\pi_+^{A\cap B}(a)$. Likewise, $\pi_-^B(a)$ exists and equals $\pi_-^{A\cap B}(a)$. □

Proposition 3.1. *If C is a poset, $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(C)$, and $A \downarrow B$ for all $(A, B) \in \mathcal{A}\times\mathcal{B}$, then $\bigcup \mathcal{A} \downarrow \bigcup \mathcal{B}$.*

Proposition 3.2. *Given a poset C and $A \subseteq B \subseteq C$, if A is a relatively complete suborder of C, then A is relatively complete suborder of B.*

Definition 3.1. Given a boolean algebra A, a long λ-approximation sequence $(M_\beta)_{\beta<\eta}$, $x \in A \cap \bigcup_{\beta<\eta} M_\beta$, and $i < \urcorner(\rho(x))$, let $\pi_+^i(x, M)$ or just $\pi_+^i(x)$ denote $\pi_+^{M_{\rho(x),i}}(x)$ if it exists; likewise let $\pi_-^i(x, M)$ or just $\pi_-^i(x)$ denote $\pi_-^{M_{\rho(x),i}}(x)$ if it exists.

Theorem 3.1. *Let A be a boolean algebra. The following are equivalent.*

1. *A has the FN.*
2. *For every long ω_1-approximation sequence $(M_\alpha)_{\alpha<|A|}$ with $A \in M_0$, for every $x \in A$, $\rho(x)$ exists and, for every $i < \urcorner(\rho(x))$, $\pi_+^i(x)$ and $\pi_-^i(x)$ exist.*
3. *There exists a long ω_1-approximation sequence $(M_\alpha)_{\alpha<|A|}$ such that, for every $x \in A$, $\rho(x)$ exists and, for every $i < \urcorner(\rho(x))$, $\pi_+^i(x)$ and $\pi_-^i(x)$ exist.*
4. *For every long ω_1-approximation sequence $(M_\alpha)_{\alpha<|A|}$ with $A \in M_0$, for every $\alpha < |A|$, and for every $i < \urcorner(\alpha)$, we have $A \cap M_\alpha \downarrow A \cap M_{\alpha,i}$ and $A \cap M'_{\alpha,i} \leq_{\mathrm{rc}} A \cap M_\alpha$.*
5. *There exists a long ω_1-approximation sequence $(M_\alpha)_{\alpha<|A|}$ such that $A \subseteq \bigcup_{\alpha<|A|} M_\alpha$ and, for all $\alpha < |A|$ and $i < \urcorner(\alpha)$, we have $A \cap M_\alpha \leq A$, $A \cap M_\alpha \downarrow A \cap M_{\alpha,i}$, and $A \cap M'_{\alpha,i} \leq_{\mathrm{rc}} A \cap M_\alpha$.*
6. *For every long ω_1-approximation sequence $(M_\alpha)_{\alpha<|A|}$ with $A \in M_0$, for every $\alpha, \beta < |A|$, and for every $i < \urcorner(\alpha)$, we have $A \cap M_\alpha \downarrow A \cap M_\beta$ and $A \cap M'_{\alpha,i} \leq_{\mathrm{rc}} A \cap M_\alpha$.*
7. *There exists a long ω_1-approximation sequence $(M_\alpha)_{\alpha<|A|}$ such that $A \subseteq \bigcup_{\alpha<|A|} M_\alpha$ and, for all $\alpha, \beta < |A|$ and $i < \urcorner(\alpha)$, we have $A \cap M_\alpha \leq A$, $A \cap M_\alpha \downarrow A \cap M_\beta$, and $A \cap M'_{\alpha,i} \leq_{\mathrm{rc}} A \cap M_\alpha$.*

Proof. (1)⇒(2). Fix M as in the hypothesis of (2). For each $x \in A$, $\rho(x)$ exists by Lemma 2.3. Each $M_{\rho(x),i}$ is an elementary substructure of $H(\mu)$ by Lemma 2.8. Also, $A \in M_{\rho(x),i}$ by Lemma 2.6. Hence, $A \cap M_{\rho(x),i} \leq_{\mathrm{rc}} A$ by Theorem 1.1. Hence, $\pi_+^i(x)$ and $\pi_-^i(x)$ exist.

(1)⇒(6). Fix M as in the hypothesis of (6). By Lemma 2.6, we have $A \in M'_{\alpha,i}$ for all $\alpha < |A|$ and $i < \urcorner(\alpha)$. Hence, by Lemma 2.8 and Theorem 1.1, we have $A \cap M'_{\alpha,i} \leq_{\mathrm{rc}} A$. Hence, $A \cap M'_{\alpha,i} \leq_{\mathrm{rc}} A \cap M_\alpha$ by Proposition 3.2. Finally, given $\alpha, \beta < |A|$, choose an FN map $f \in M_0$. By Lemma 2.6, M_α and M_β are f-closed; hence, $A \cap M_\alpha \downarrow A \cap M_\beta$.

(2)⇒(3), (4)⇒(5), and (6)⇒(7). Choose μ large enough that $A \in H(\mu)$. By Lemma 2.1, there is a long ω_1-approximation sequence $(M_\alpha)_{\alpha<|A|}$ with $A \in M_0$. By Lemma 2.6, we have $A \cap M_\alpha \leq A$ for all $\alpha < |A|$. By Lemma 2.3, we have $A \subseteq \bigcup_{\alpha<|A|} M_\alpha$.

(6)⇒(4) and (7)⇒(5). Apply Proposition 3.1.

(5)⇒(3). By Lemma 3.1, $\pi_+^{M_{\alpha,i}}$ and $\pi_-^{M_{\alpha,i}}$ are well-defined on all of $A \cap M_\alpha$.

(3)⇒(1). This is implicit in the author's proof of Corollary 3.4 of [8], but we include a proof here for completeness. For each $\alpha < |A|$, choose a well-ordering $\sqsubseteq_\alpha$ of $\{x \in A : \rho(x) = \alpha\}$ with length at most ω. Set $\sqsubseteq = \bigcup_{\alpha<|A|} \sqsubseteq_\alpha$. Recursively define $f\colon A \to [A]^{<\aleph_0}$ by

$$f(x) = \{y : y \sqsubseteq x\} \cup \left(\bigcup_{i<\urcorner(\rho(x))} \left(f(\pi_+^i(x)) \cup f(\pi_-^i(x))\right) \right).$$

Suppose $x \leq_A y$. We verify that $S = [x,y] \cap f(x) \cap f(y)$ is nonempty by induction on $\max\{\rho(x), \rho(y)\}$. If $\rho(x) = \rho(y)$, then $x \sqsubseteq y$, in which case $x \in S$, or $y \sqsubseteq x$, in which case $y \in S$. If $\rho(x) < \rho(y)$, then $x \in M_{\rho(y),i}$ for some i, in which case $[x, \pi_-^i(y)] \cap f(x) \cap f(\pi_-^i(y))$ is a nonempty subset of S. If $\rho(y) < \rho(x)$, then $y \in M_{\rho(x),i}$ for some i, in which case $[\pi_+^i(x), y] \cap f(\pi_+^i(x)) \cap f(y)$ is a nonempty subset of S. □

Lemma 3.2. *Given boolean algebras $A \leq C$ and $B \leq_{\mathrm{rc}} C$, the following are equivalent.*

(1) $A \downarrow B$.
(2) $\pi_+^B[A] \subseteq A$.
(3) $\pi_-^B[A] \subseteq A$.

Proof. (1)⇒(2). Given $a \in A$, we have $a \leq \pi_+^B(a) \in B$, so there exists $b \in [a, \pi_+^B(a)] \cap A \cap B$. However, by definition of π_+^B, we must have have $b = \pi_+^B(a)$.

(2)⇒(1). Given $a \in A$ and $b \in B$, if $a \leq b$, then $\pi_+^B(a) \in [a,b] \cap A \cap B$; if $b \leq a$, then $\pi_-^B(a) \in [b,a] \cap B$ and $\pi_-^B(a) = -\pi_+^B(-a) \in A$.

(2)⇔(3). $\pi_-^B(\bullet) = -\pi_+^B(-\bullet)$ and $\pi_+^B(\bullet) = -\pi_-^B(-\bullet)$. □

Definition 3.2.

- A *partial algebra* is a pair of the form $(U, \mathcal{F})$ where U is a set (called the universe of $(U,\mathcal{F})$) and $\mathcal{F}$ is a set of functions such that, for each $f \in \mathcal{F}$, there exists $n < \omega$ such that $\mathrm{dom}(f) \subseteq A^n$. If every $\mathrm{dom}(f)$ is of the form A^n, then we say that $(U, \mathcal{F})$ is an *algebra.*
- A partial algebra $(U,\mathcal{F})$ is a *subalgebra* of a partial algebra $(V, \mathcal{G})$ if $U \subseteq V$, $\mathcal{F} = \{g \restriction U^{<\omega} : g \in \mathcal{G}\}$, and $\bigcup_{g\in\mathcal{G}} g[U^{<\omega}] \subseteq U$.
- A partial algebra $(U,\mathcal{F})$ is *locally finite* if, for every finite $A \subseteq U$, there exists a finite $B \subseteq U$ such that $A \subseteq B$ and $(B, \{f \restriction B : f \in \mathcal{F}\})$ is a subalgebra of $(U, \mathcal{F})$.
- We say that a partial algebra $(V,\mathcal{G})$ *expands* a partial algebra $(U,\mathcal{F})$ if $U = V$ and $\mathcal{F} \subseteq \mathcal{G}$.
- Given a long λ-approximation sequence $(M_i)_{i<\eta}$ and a boolean algebra A, let the *M-expansion of A*, written $A[M]$, denote the expansion of A resulting from adding the functions in the set $\bigcup_{i<\omega}\{\pi_+^i, \pi_-^i\}$.

Theorem 3.2. *If A is a boolean algebra with the SFN, $(M_\alpha)_{\alpha<|A|}$ is a long ω_1-approximation sequence, and $A \in M_0$, then $A[M]$ is locally finite.*

Proof. Let $\mathcal{C} \in M_0$ be a pairwise commuting cofinal family of finite subalgebras of A. It suffices to show that, for every $F \in \mathcal{C}$, (an expansion of) F is a subalgebra of $A[M]$. Let $x \in F \in \mathcal{C}$, $\alpha = \rho(x)$, and $i < \daleth(\alpha)$. By Theorem 3.1, $\pi^i_\pm(x)$ are well-defined. By Lemmas 2.6 and 2.8, $\mathcal{C} \in M_{\alpha,i} \prec H(\mu)$, so we may choose $G \in M_{\alpha,i} \cap \mathcal{C}$ such that $\pi^i_\pm(x) \in G$. Since $G \in M_{\alpha,i}$ and G is finite, $G \subseteq M_{\alpha,i}$; hence, $\pi^i_\pm(x) \in G$ implies $\pi^i_\pm(x) = \pi^G_\pm(x)$. By Lemma 3.2, $\pi^G_\pm(x) \in F$. □

The proof of Theorem 3.2 implicitly shows much more. Indeed, we can expand $A[M]$ by adding every function of the form $\pi^N_\pm$ where $\mathcal{C} \in N \prec H(\mu)$, yet still obtain a locally finite partial algebra. However, local finiteness of $A[M]$ is strong enough for our purposes. As we shall show in Section 5, it is strictly stronger than the FN.

Question 3.1. If A is a boolean algebra with the FN, $(M_\alpha)_{\alpha<|A|}$ is a long ω_1-approximation sequence, $A \in M_0$, and $A[M]$ is locally finite, then does A have the SFN?

We do not know the answer to the above question. However, we will point out that if we broaden our consideration to arbitrary expansions of boolean algebras, then characterizations of the FN and SFN are apparently easier to obtain.

Definition 3.3. Call an FN map f *transitive* if $f(y) \subseteq f(x)$ for all $x \in \mathrm{dom}(f)$ and $y \in f(x)$.

Lemma 3.3. *If A has the FN, then A has a transitive FN map.*

Proof. Construct an FN map f as in the proof of (4)⇒(1) in Theorem 3.1, except use the following recursive definition of f:

$$f(x) = \{x\} \cup \left(\bigcup_{y \sqsubset x} f(y) \right) \cup \left(\bigcup_{i < \daleth(\rho(x))} \left(f(\pi^i_+(x)) \cup f(\pi^i_-(x)) \right) \right).$$

The proof of (4)⇒(1) in Theorem 3.1 still works *verbatim*, but now f is also transitive. □

Lemma 3.4. *If A is a boolean algebra, $\mathcal{B}$ is a pairwise commuting family of relatively complete subalgebras of A, and $B_0, B_1 \in \mathcal{B}$, then $\mathcal{B} \cup \{B_0 \cap B_1\}$ is pairwise commuting.*

Proof. Let $C \in \mathcal{B}$. Suppose that $x \in B_0 \cap B_1$, $y \in C$, and $x \le y$. By symmetry, it suffices to show that $[x, y] \cap B_0 \cap B_1 \cap C$ is nonempty. Let $z = \pi^C_+(x)$, which is in $[x, y] \cap C$. By Lemma 3.2, z is also in each of B_0 and B_1. □

Definition 3.4.

- Given a partial algebra B, call a subalgebra C of B *cyclic* if, for some $x \in C$, C is the smallest subalgebra of B that contains $\{x\}$.
- Given a boolean algebra A, we say that a partial algebra B is *strongly A-commuting* if B has the same universe as A, B is locally finite, and, for all cyclic subalgebras F and G of B, we have $F \perp G$ as suborders of $(A, \le_A)$.

Theorem 3.3. *Given a boolean algebra A,*

- *A has the FN if and only if there is a strongly A-commuting algebra;*
- *A has the SFN if and only if there is a strongly A-commuting algebra expanding A.*

Moreover, the above is true if we replace "strongly A-commuting algebra" with "strongly A-commuting partial algebra."

Proof. If f is a transitive FN map on A, then, letting $(f_n(x))_{n<\omega}$ surject from ω to $f(x)$ for each $x \in A$, the algebra B with universe A and set of functions $\{f_n : n < \omega\}$ is strongly A-commuting.

Conversely, given a strongly A-commuting partial algebra B, construct an FN map f by letting $f(x)$ be the minimal subalgebra of B containing $\{x\}$, for each $x \in A$.

Suppose $\mathcal{C}$ is a pairwise commuting cofinal family of finite subalgebras of A. By Lemma 3.4, we may assume that $\mathcal{C}$ is closed with respect to pairwise intersection. For each $x \in A$, let $\mathcal{C}(x)$ denote the smallest element of $\mathcal{C}$ that contains $\{x\}$; let $(f_n(x))_{n<\omega}$ surject from ω to $\mathcal{C}(x)$. The expansion of A formed by the adding the functions from $\{f_n : n < \omega\}$ is strongly A-commuting.

Conversely, suppose that B is a strongly A-commuting expansion of A. Let $\mathcal{C}$ denote the set of finite subalgebras of B. Since B is locally finite, $\mathcal{C}$ is a cofinal family of finite subalgebras of A (provided we identify each $C \in \mathcal{C}$ with the subalgebra of A that has the same universe). Moreover, by Proposition 3.1, $\mathcal{C}$ is pairwise commuting. $\square$

Observe that adapting the proof of Lemma 3.3 to build a strongly A-commuting expansion of A would require not only that $A[M]$ be locally finite, but also that $A[M]$ remain locally finite after adding a partial function that maps each $x \in A$ to its immediate $\sqsubset$-predecessor, if one exists.

We shall need the next lemmas in Section 5.

Lemma 3.5. *If $A \leq_{\mathrm{rc}} B$, $a \in A$, $b \in B$, and $\pi^A_+(b) = x$, then $\pi^A_+(a \wedge b) = a \wedge x$.*

Proof.

$$\begin{aligned} a \wedge b \leq y \in A &\Rightarrow b \leq y \vee -a \in A \\ &\Rightarrow b \leq x \leq y \vee -a \\ &\Rightarrow a \wedge b \leq a \wedge x \leq y \end{aligned}$$ $\square$

Definition 3.5. Given a boolean algebra A, a long λ-approximation sequence $(M_\alpha)_{\alpha<\eta}$, and $x \in A$, let $\sigma^{\varnothing}_+(x, M) = x$ and, for all $(t_0, \ldots, t_n) \in \omega^{<\omega}$, let

$$\sigma^t_+(x, M) = (\pi^{t_n}_+ \circ \pi^{t_{n-1}}_+ \circ \cdots \circ \pi^{t_0}_+)(x, M)$$

if the righthand side exists. Let $\varsigma_+(x, M)$ denote the set of all $\sigma^t_+(x, M)$ that exist. Likewise define $\sigma^t_-(x, M)$ and $\varsigma_-(x, M)$. We may suppress the dependence of $\varsigma_\pm$ and $\sigma^t_\pm$ on M when convenient.

Observe that if s is a strict initial segment of t and $\sigma^t_+(x)$ exists, then $t_{|s|} < \urcorner(\rho(\sigma^s_+(x)))$ and $\rho(\sigma^t_+(x)) < \rho(\sigma^s_+(x))$. Hence, by König's Lemma, $\varsigma_+(x)$ is finite; likewise, $\varsigma_-(x)$ is finite.

Lemma 3.6. *Suppose we have a boolean algebra A and a long λ-approximation sequence $(M_\alpha)_{\alpha<\eta}$ such that $\pi^i_\pm(x)$ exist for all $x \in A \cap \bigcup_{\alpha<\eta} M_\alpha$ and all $i <$*

$\daleth(\rho(x))$. *Then, for every $B \leq A[M]$ where B is of the form $A \cap \bigcup_{\alpha \in I} M_\alpha$, we have $\pi_+^B(x) = \bigwedge(B \cap \varsigma_+(x))$ and $\pi_-^B(x) = \bigvee(B \cap \varsigma_-(x))$ for all $x \in A \cap \bigcup_{\alpha<\eta} M_\alpha$.*

Proof. Suppose that $x \in A \cap \bigcup_{\alpha<\eta} M_\alpha$ and $x \leq y \in B$. By symmetry, it suffices to show that $x_+^B \leq y$ where $x_+^B = \bigwedge(B \cap \varsigma_+(x))$. Proceed by induction on $\max\{\rho(x), \rho(y)\}$. Choose $\alpha \in I$ such that $y \in M_\alpha$. By Lemma 2.11, $M_{\rho(y)} \subseteq M_\alpha$; hence, $A \cap M_{\rho(y)} \subseteq B$. If $\rho(x) = \rho(y)$, then $x \in B$, in which case $x_+^B = x \leq y$. If $\rho(x) < \rho(y)$, then $x \in M_{\rho(y),i}$ for some i, in which case $x \leq \pi_-^i(y) \in B$ and $\rho(\pi_-^i(y)) < \rho(y)$; by induction, $x_+^B \leq \pi_-^i(y) \leq y$. If $\rho(y) < \rho(x)$, then $y \in M_{\rho(x),i}$ for some i, in which case $z \leq y$ and $\rho(z) < \rho(x)$ where $z = \pi_+^i(x)$; by induction, $z_+^B \leq y$; hence, $x_+^B \leq z_+^B \leq y$ because $\varsigma_+(z) \subseteq \varsigma_+(x)$. □

Corollary 3.1. *Suppose we have a boolean algebra A and a long λ-approximation sequence $(M_\alpha)_{\alpha<\eta}$ such that $\rho(x)$ and $\pi_\pm^i(x)$ exist for all $x \in A$ and all $i < \daleth(\rho(x))$. Further suppose that ν is a regular uncountable cardinal, $A, M \in P \prec H(\nu)$, and $\lambda \cap P \in \lambda + 1$. Then, for all $x \in A \setminus P$,*

$$\pi_+^P(x) = \bigwedge_{i<\daleth(\rho(x,M))} \pi_+^P(\pi_+^i(x,M)) \text{ and}$$
$$\pi_-^P(x) = \bigvee_{i<\daleth(\rho(x,M))} \pi_-^P(\pi_-^i(x,M)).$$

4. Embeddings and colimits

In this section, we collect some facts and specify some notation concerning various colimits and various classes of boolean embeddings. We refer the reader to [4] and [6] for additional background information.

Definition 4.1.
- Say that a sequence $(F_i)_{i\in I}$ of subalgebras of a fixed boolean algebra is *independent* if, for all finite $J \subseteq I$ and all $x \in \prod_{j\in J} F_j$, if $\bigwedge_{j\in J} x(j) = 0$, then $x(j) = 0$ for some $j \in J$.
- Say that a sequence $(x_i)_{i\in I}$ of elements of a fixed boolean algebra is *independent* if $(\{x_i, -x_i, 0, 1\})_{i\in I}$ is independent.
- Say that a boolean algebra F is *free* if it is generated by the range of an independent sequence of elements of F.

Fix once and for all a countably infinite free boolean algebra Fr_ω and an independent sequence $(\mathrm{fr}_n)_{n<\omega}$ generating Fr_ω such that Fr_ω and fr are definable in $H(\aleph_1)$ without parameters. For each $S \subset \omega$, let Fr_S denote the subalgebra of Fr_ω generated by $\{\mathrm{fr}_n : n \in S\}$.

Definition 4.2.
- A *boolean embedding* is an injective boolean homomorphism.
- Given two boolean algebras C_0 and C_1, a *coproduct* of C_0 and C_1 is a boolean algebra $C_0 \oplus C_1$ with boolean embeddings $\oplus_0 : C_0 \to C_0 \oplus C_1$ and $\oplus_1 : C_1 \to C_0 \oplus C_1$ such that $\oplus_0[C_0]$ and $\oplus_1[C_1]$ are independent and $\bigcup_{i<2} \oplus_i[C_i]$ generates $C_0 \oplus C_1$. These embeddings are called *cofactor maps*.

Coproducts always exist uniquely up to isomorphism.

Definition 4.3.

- Say that a boolean embedding $f\colon A \to B$ is *free* if there is an infinite free boolean algebra F and a coproduct $A \oplus F$ such that $A \oplus F = B$ and $\oplus_0 = f$.
- If id_A is free embedding from A to B, then we say that B is a *free extension* of A and write $A \leq_{\mathrm{free}} B$.
- Given boolean algebras $A \leq B$, we say that A *splits* in B if every ultrafilter of A extends to at least two ultrafilters of B. We say that A *splits perfectly* in B if, for all finite $F \subseteq B$, the subalgebra generated by $A \cup F$ splits in B.
- We say that a boolean embedding $f\colon A \to B$ *splits perfectly* if $f[A]$ splits perfectly in B.

Every free embedding is relatively complete and splits perfectly. Conversely, we have *Sirota's Lemma* [12], *i.e.*, if $f\colon A \to B$ is relatively complete and perfectly splitting and B is generated by $f[A] \cup C$ for some countable C, then f is free. Also note that the classes of relatively complete, perfectly splitting, and free embeddings are each closed with respect to composition. Moreover, for any composite boolean embedding $f \circ g$, if f is perfectly splitting, then so is $f \circ g$.

Definition 4.4.

- A *quotient* of a boolean algebra A with respect to an ideal I is a boolean algebra B with a surjective homomorphism $f\colon A \to B$ with kernel I; f is called the *quotient map* and $f(x)$ may be denoted by x/I.
- Given boolean embeddings $f\colon C \to A$ and $g\colon C \to B$, define a *pushout* $A \underset{C}{\boxplus} B$ of f and g to be a quotient of a coproduct $A \oplus B$ with respect to the ideal I generated by $\{\oplus_0(f(c)) \wedge \oplus_1(g(-c)) : c \in C\}$. Thus, $A \underset{C}{\boxplus} B$ is a colimit of the diagram formed by f and g.
- Given f and g as above such that also $f = g = \mathrm{id}_{A\cap B}$, let $A \boxplus B$ more specifically denote a pushout of f and g such that $\oplus_0(a)/I = a$ and $\oplus_1(b)/I = b$ for all $a \in A$ and $b \in B$.

If A and B are boolean algebras such that their intersection $A \cap B$ is also a common subalgebra, then $A \boxplus B$ exists as above and is characterized up to isomorphism as a boolean algebra D in which A and B are commuting subalgebras and $A \cup B$ generates D.

Lemma 4.1. *If $A = C_0 \cap C_1$ and, for each $i < 2$, we have $C_i = A \oplus B_i$ with cofactor maps $\oplus_0 = \mathrm{id}_A$ and $\oplus_1 = \mathrm{id}_{B_i}$, then (A, B_0, B_1) is independent in $C_0 \boxplus C_1$*

Proof. Suppose that $a \in A$, $b_0 \in B_0$, $b_1 \in B_1$, and $a \wedge b_0 \wedge b_1 = 0$ in $C_0 \boxplus C_1$. Then $a \wedge b_0 \leq \hat{a} \leq -b_1$ for some $\hat{a} \in A$ because $C_0 \downarrow C_1$. Hence, $\hat{a} \wedge b_1 = 0$. Since (A, B_1) is independent, $\hat{a} = 0$ or $b_1 = 0$. If $\hat{a} = 0$, then $a \wedge b_0 = 0$, in which case $a = 0$ or $b_0 = 0$ because (A, B_0) is independent. Thus, $a = 0$, $b_0 = 0$, or $b_1 = 0$. □

Definition 4.5. Given a (nonempty) directed set $\mathcal{D}$ of boolean algebras such that $A \subseteq B$ implies $A \leq B$ for all $A, B \in \mathcal{D}$, endow the union $\bigcup \mathcal{D}$ of (the universes of) the algebras in $\mathcal{D}$ with the unique algebraic operations that make the inclusions $(\mathrm{id}_A\colon A \to \bigcup \mathcal{D})_{A\in\mathcal{D}}$ a colimit of the inclusions $(\mathrm{id}_A\colon A \to B)_{\{A\subseteq B\}\subseteq\mathcal{D}}$, namely, $\wedge_{\bigcup\mathcal{D}} = \bigcup_{A\in\mathcal{D}} \wedge_A$, $\vee_{\bigcup\mathcal{D}} = \bigcup_{A\in\mathcal{D}} \vee_A$, $-_{\bigcup\mathcal{D}} = \bigcup_{A\in\mathcal{D}} -_A$, $0_{\bigcup\mathcal{D}} = 0_B$, and $1_{\bigcup\mathcal{D}} = 1_B$ for some $B \in \mathcal{D}$.

5. Proof of main theorem

By Theorem 3.2, it is sufficient to construct a boolean algebra Ω of size $\aleph_2$ such that Ω has the FN, but $\Omega[N]$ is not locally finite for some long ω_1-approximation sequence $(N_\alpha)_{\alpha<\omega_2}$ with $\Omega \in N_0$. We will construct in parallel a sequence $(A_\alpha)_{\alpha<\omega_2}$ of countable boolean algebras and a long ω_1-approximation sequence $(M_\alpha)_{\alpha<\omega_2}$ such that, for all $\alpha < \omega_2$, we have

$$A_\alpha \Subset M_\alpha \prec H(\aleph_2), \tag{1}$$

$$A_\alpha \cap \bigcup_{\beta<\alpha} M_\beta = \bigcup_{i<\urcorner(\alpha)} A'_{\alpha,i}, \text{ and} \tag{2}$$

$$\forall i < \urcorner(\alpha)\ \ A'_{\alpha,i} \leq A_\alpha \tag{3}$$

where $A'_{\alpha,i} = \bigcup\{A_\beta : \beta \in J'_i(\alpha)\}$ and $S \Subset T$ means that S is a coinfinite subset of T. Also define $A_{\alpha,i} = \bigcup\{A_\beta : \beta \in J_i(\alpha)\}$.

Claim 5.1. Given A and M as above, $\alpha, \beta < \omega_2$, and $i < \urcorner(\alpha)$, we have that

- $A_\alpha \cap A_\beta = A_\alpha \cap M_\beta$,
- $A_\alpha \subseteq A_\beta$ if $M_\alpha \subseteq M_\beta$, and
- $A'_{\alpha,i} = A_\alpha \cap A_{\alpha,i}$.

Proof. For the first subclaim, we may assume we are not in the trivial case $\alpha = \beta$. Since A_β is countable, (1) implies $A_\beta \subseteq M_\beta$. Therefore, $A_\alpha \cap A_\beta \subseteq A_\alpha \cap M_\beta$. To prove the converse inclusion, suppose $x \in A_\alpha \cap M_\beta$. If $\beta < \alpha$, then, by (2), $x \in A_\gamma$ for some $\gamma < \alpha$; we may inductively assume $A_\gamma \cap M_\beta = A_\gamma \cap A_\beta$, in which case $x \in A_\beta$. If $\alpha < \beta$, then $\rho(x) \in M_\beta \cap I_j(\beta) \subseteq J'_j(\beta)$ for some $j < \urcorner(\beta)$; hence, by (2) and (3), $x \in A_{\rho(x)} \subseteq A'_{\beta,j} \subseteq A_\beta$. Thus, $A_\alpha \cap M_\beta \subseteq A_\alpha \cap A_\beta$. Therefore, $A_\alpha \cap A_\beta = A_\alpha \cap M_\beta$.

We obtain the third subclaim from the first subclaim and from (3):

$$\begin{aligned} A_\alpha \cap A_{\alpha,i} = A_\alpha \cap \bigcup_{\beta\in J_i(\alpha)} A_\beta = A_\alpha \cap \bigcup_{\beta\in J_i(\alpha)} M_\beta = A_\alpha \cap M_\alpha \cap \bigcup_{\beta\in J_i(\alpha)} M_\beta \\ = A_\alpha \cap \bigcup_{\beta\in J'_i(\alpha)} M_\beta = A_\alpha \cap \bigcup_{\beta\in J'_i(\alpha)} A_\beta = A_\alpha \cap A'_{\alpha,i} = A'_{\alpha,i}. \end{aligned}$$

Also, the second subclaim holds because if $\delta < \omega_2$ and $M_\alpha \subseteq M_\delta$, then $A_\alpha \subseteq A_\delta$ because $A_\alpha \cap A_\delta = A_\alpha \cap M_\delta \supseteq A_\alpha \cap M_\alpha = A_\alpha$. □

By the above claim, since $\mathcal{I}_0(\omega_2)$ is directed, letting $\Omega = A_{\omega_2,0} = \bigcup_{\alpha<\omega_2} A_\alpha$, we obtain a boolean algebra of size at most $\aleph_2$ such that, for all $\alpha < \omega_2$, $\Omega \cap M_\alpha = A_\alpha \leq \Omega$ and, for all $i < \urcorner(\alpha)$, $\Omega \cap M_{\alpha,i} = A_{\alpha,i}$ and $\Omega \cap M'_{\alpha,i} = A'_{\alpha,i}$. Therefore, by Theorem 3.1, Ω will have the FN if $A'_{\alpha,i} \leq_{\mathrm{rc}} A_\alpha$ and $A_\alpha \downarrow A_\beta$ as suborders of Ω for all $\alpha, \beta < \omega_2$ and $i < \urcorner(\alpha)$. Therefore, Ω will have the FN if we have the following for all $\alpha < \omega_2$.

$$\forall i < \urcorner(\alpha)\ \ A'_{\alpha,i} \leq_{\mathrm{free}} A_\alpha. \tag{4}$$

$$\forall M_\beta, M_\gamma \in M_\alpha\ \ A_\beta \downarrow A_\gamma \text{ as suborders of } A_\alpha. \tag{5}$$

Note that (4) implies (3).

At stage 0, let $A_0 = \mathrm{Fr}_\omega \in M_0$. At nonzero stages $\alpha < \omega_2$, select a countable $M_\alpha \prec H(\aleph_2)$ such that $(A_\beta, M_\beta)_{\beta<\alpha} \in M_\alpha$. If $\urcorner(\alpha) = 1$, then let A_α be a coproduct

$A'_{\alpha,0} \oplus \mathrm{Fr}_\omega$ such that $A'_{\alpha,0} \le A_\alpha \Subset M_\alpha$ and $A_\alpha \setminus A'_{\alpha,0}$ is disjoint from $\bigcup_{\beta<\alpha} M_\beta$. Clearly, (1), (2), and (4) are preserved. Since $\mathcal{J}'_0(\alpha)$ is directed, (5) is preserved too.

Now suppose that $\daleth(\alpha) = 2$. Let $A'_{\alpha,2} = \bigcap_{j<2} A'_{\alpha,j}$. By Lemma 2.10, $A'_{\alpha,2}$ is a directed union of common subalgebras of $A'_{\alpha,0}$ and $A'_{\alpha,1}$.

Claim 5.2. $A'_{\alpha,2} \le_{\mathrm{rc}} A'_{\alpha,i}$ for each $i < 2$.

Proof. Fix $i < 2$. By Lemma 2.12, for each $\beta \in J'_i(\alpha)$ and $j < \daleth(\beta)$, $A'_{\alpha,i} \cap A_{\beta,j} = \bigcup\{A_\gamma : \gamma \in U\}$ for some $U \subseteq J'_i(\alpha)$. By the inductive assumption of (5) for all stages before α and by the directedness of $\{A_\delta : \delta \in J'_i(\alpha)\}$, $A_\gamma \downarrow A_\beta$ for all $\gamma \in U$. Hence, by Proposition 3.1, $A'_{\alpha,i} \cap A_{\beta,j} \downarrow A_\beta$. By the inductive assumption of (4) for all stages before α, we have $A'_{\alpha,i} \cap A_{\beta,j} \cap A_\beta = A'_{\beta,j} \le_{\mathrm{rc}} A_\beta$. Hence, by Lemma 3.1, $\pi_\pm^{A'_{\alpha,i}\cap A_{\beta,j}}(x)$ exist for all $x \in A_\beta$. Therefore, applying Lemma 3.6 with $A'_{\alpha,i}$ in place of A and $A'_{\alpha,2}$ in place of B, we have $A'_{\alpha,2} \le_{\mathrm{rc}} A'_{\alpha,i}$. □

Claim 5.3. $A'_{\alpha,2} \le_{\mathrm{free}} A'_{\alpha,i}$ for each $i < 2$.

Proof. Fix $i < 2$. By Sirota's Lemma and the previous claim, it suffices to show that $A'_{\alpha,2}$ splits perfectly in $A'_{\alpha,i}$. Let U be an ultrafilter of $A'_{\alpha,2}$ and F be a finite subset of $A'_{\alpha,i}$. Applying Lemma 2.9, there exists $\beta \in I'_i(\alpha) \setminus J_{1-i}(\alpha)$. Since $\mathcal{I}'_i(\alpha)$ is directed and $\mathcal{J}_{1-i}(\alpha)$ is downward closed in $\{M_\beta : \beta < \alpha\}$, we may choose $\beta \in I'_i(\alpha) \setminus J_{1-i}(\alpha)$ such that $F \subseteq A_\beta$. By Lemmas 2.12 and 2.13, $A_\beta \cap A'_{\alpha,2} \subseteq A'_{\beta,j}$ for some $j < \daleth(\beta)$. Let $B \le A_\beta$ denote the subalgebra generated by $A'_{\beta,j} \cup F$. Extend $U \cap A_\beta$ to an ultrafilter V of B. Since $A'_{\beta,j} \le_{\mathrm{free}} A_\beta$ by (4) for stage β, B splits in A_β. So, choose $y \in A_\beta$ and ultrafilters $V_\pm$ of A_β that respectively extend $V \cup \{\pm y\}$. By (5) for stages before α and by Proposition 3.1, $A'_{\alpha,2} \downarrow A_\beta$ as suborders of $A'_{\alpha,i}$, so both of $U \cup V_\pm$ extend to ultrafilters of $A'_{\alpha,i}$. Thus, $A'_{\alpha,2}$ splits perfectly in $A'_{\alpha,i}$. □

Choose $B_\alpha = A'_{\alpha,0} \boxplus A'_{\alpha,1}$ such that $B_\alpha \Subset M_\alpha$ and $B_\alpha \setminus (A'_{\alpha,0} \cup A'_{\alpha,1})$ is disjoint from $\bigcup_{\beta<\alpha} M_\beta$.

Claim 5.4. (5) will be preserved if $B_\alpha \le A_\alpha$.

Proof. Suppose that $x_i \in A_{\beta_i}$ and $M_{\beta_i} \in M_\alpha$ for each $i < 2$, and that $x_0 \le x_1$ in A_α. It suffices to find $w \in [x_0, x_1] \cap M_{\beta_0} \cap M_{\beta_1}$. Let $\beta_i \in J'_{j_i}$ for each $i < 2$. We inductively assume that (5) holds for all stages before α. Therefore, since $\mathcal{J}'_0(\alpha)$ and $\mathcal{J}'_1(\alpha)$ are directed, if $j_0 = j_1$, then w as above exists. So, assume that $j_0 \ne j_1$. By symmetry, we may assume that $j_0 = 0$. Assuming $B_\alpha \le A_\alpha$, we have $A'_{\alpha,0} \downarrow A'_{\alpha,1}$ as suborders of A_α. Hence, we may choose $y \in A'_{\alpha,0} \cap A'_{\alpha,1} \cap [x_0, x_1]$. Choose $\gamma \in K'_2(\alpha)$ such that $y \in M_\gamma$. We may choose $z_0 \in [x_0, y] \cap A_{\beta_0} \cap A_\gamma$ and $z_1 \in [y, x_1] \cap A_{\beta_1} \cap A_\gamma$, again because $\mathcal{J}'_0(\alpha)$ and $\mathcal{J}'_1(\alpha)$ are directed. For each $i < 2$, let $\delta_i = \rho(z_i, M \restriction \alpha)$; we then have $z_i \in M_{\delta_i} \subseteq M_{\beta_i} \cap M_\gamma$ by Lemma 2.11. Since $\mathcal{K}'_2(\alpha)$ is also directed, we may choose $w \in [z_0, z_1] \cap A_{\delta_0} \cap A_{\delta_1}$. Thus, $w \in [x_0, x_1] \cap A_{\beta_0} \cap A_{\beta_1}$. □

Given Claim 5.3, we may choose, for each $i < 2$, cofactor maps $\oplus_0 = \mathrm{id}\colon A'_{\alpha,2} \to A'_{\alpha,i}$ and $\oplus_1 = \zeta_{\alpha,i}\colon \mathrm{Fr}_\omega \to A'_{\alpha,i}$. For each $i < 2$, let $B_{\alpha,i} = \mathrm{ran}(\zeta_{\alpha,i})$ and $b^n_{\alpha,i} = \zeta_{\alpha,i}(\mathrm{fr}_n)$ for each $n < \omega$. Choose $C_\alpha = B_\alpha \oplus \mathrm{Fr}_2$ such that $C_\alpha \Subset M_\alpha$, that $\oplus_0 = \mathrm{id}_{B_\alpha}$, and that $C_\alpha \setminus B_\alpha$ is disjoint from $\bigcup_{\beta<\alpha} M_\beta$. For each $i < 2$, let

$h_{\alpha,i} = \eta_\alpha(\mathrm{fr}_i)$ where η_α is the cofactor map $\oplus_1\colon \mathrm{Fr}_2 \to C_\alpha$. Let $H_\alpha = \mathrm{ran}(\eta_\alpha)$. By Lemma 4.1, $(A'_{\alpha,2}, B_{\alpha,0}, B_{\alpha,1})$ is independent; hence, so is $(A'_{\alpha,2}, B_{\alpha,0}, B_{\alpha,1}, H_\alpha)$. For each $n < \omega$, let $e^n_{\alpha,0} = b^n_{\alpha,0} \wedge b^n_{\alpha,1} \wedge h_{\alpha,0}$ and $e^n_{\alpha,1} = b^{n+1}_{\alpha,0} \wedge b^n_{\alpha,1} \wedge h_{\alpha,1}$; let I_α be the ideal of C_α generated by $\{e^n_{\alpha,i} : (i,n) \in 2 \times \omega\}$.

Claim 5.5. $I_\alpha \cap (B_\alpha \cup H_\alpha) = \{0\}$.

Proof. Let $1 \leq m < \omega$, let $e = \bigvee_{(i,n)\in 2\times m} e^n_{\alpha,i}$, and let J be the ideal generated by e. We will show that $J \cap (B_\alpha \cup H_\alpha) = \{0\}$. First, if $0 < h \in H_\alpha$, then $h \not\leq e$ because $e \wedge c = 0 < c \leq h$ where $c = h \wedge -b^0_{\alpha,1}$. Second, if $0 < b \in B_\alpha$, then $b \not\leq e$ because $e \wedge c = 0 < c \leq b$ where $c = b \wedge -h_{\alpha,0} \wedge -h_{\alpha,1}$. □

By the above claim, we may choose a quotient $D_\alpha = C_\alpha/I_\alpha$ such that $x/I_\alpha = x$ for all $x \in B_\alpha \cup H_\alpha$. Choose D_α such that also $D_\alpha \Subset M_\alpha$ and $D_\alpha \setminus (B_\alpha \cup H_\alpha)$ is disjoint from $\bigcup_{\beta<\alpha} M_\beta$. Note that $B_\alpha \not\leq_{\mathrm{rc}} D_\alpha$ because, for example, $\pi^{B_\alpha}_+(h_\alpha)$ does not exist because $h_{\alpha,0} \leq_{D_\alpha} \bigwedge_{n<m} -(b^n_{\alpha,0} \wedge b^n_{\alpha,1})$ for all $m < \omega$. However, we still have the following.

Claim 5.6. For each $i < 2$, $A'_{\alpha,i} \leq_{\mathrm{rc}} D_\alpha$. In particular, $\pi^{A'_{\alpha,0}}_+$ and $\pi^{A'_{\alpha,1}}_+$ satisfy (6) for all $n < \omega$ and $x \in D_\alpha$ of the forms below.

(6)

x	$\pi^{A'_{\alpha,0}}_+(x)$	$\pi^{A'_{\alpha,1}}_+(x)$
$b^n_{\alpha,0} \wedge h_{\alpha,0}$	$b^n_{\alpha,0}$	$-b^n_{\alpha,1}$
$b^n_{\alpha,1} \wedge h_{\alpha,1}$	$-b^{n+1}_{\alpha,0}$	$b^n_{\alpha,1}$

Proof. For this proof our notation will suppress the dependence on α. Every nonzero element of C is a finite nonempty join of elements of the form $x = a \wedge b_0 \wedge b_1 \wedge h$ where $a \in A'_2$, h is of the form $\pm h_0 \wedge \pm h_1$, and each b_i is $\bigwedge_{n\in P_i} b^n_i \wedge \bigwedge_{n\in Q_i} -b^n_i$ where P_i and Q_i are each (possibly empty) finite subsets of ω. (Our convention is that $\bigwedge \varnothing = 1$.) In general, $\pi^{A'_i}_+(y \vee z) = \pi^{A'_i}_+(y) \vee \pi^{A'_i}_+(z)$ and $\pi^{A'_i}_-(y) = -\pi^{A'_i}_+(-y)$ if the righthand sides exist. Moreover, by Lemma 3.5,

$$\pi^{A'_i}_+((a \wedge b_i \wedge b_{1-i} \wedge h)/I) = a \wedge b_i \wedge \pi^{A'_i}_+((b_{1-i} \wedge h)/I)$$

if the righthand side exists.

Let $x = b_{1-i} \wedge_C h$ where b_{1-i} and h are as above. We will show that each $\pi^{A'_i}_+(x/I)$ exists and equals $\tau_i(x)$ where $\tau_i(x) = \bigwedge_{n\in T_i} -b^n_i$ where T_i is as in (7) below, which uses shift operator notation $S\triangleright = \{\beta + 1 : \beta \in S\}$ and $S\triangleleft = \{\beta : \beta + 1 \in S\}$ for sets of ordinals.

(7)

h			T_0	T_1
$-h_0$	$\wedge$	$-h_1$	$\varnothing$	$\varnothing$
h_0	$\wedge$	$-h_1$	P_1	P_0
$-h_0$	$\wedge$	h_1	$P_1\triangleright$	$P_0\triangleleft$
h_0	$\wedge$	h_1	$P_1 \cup (P_1\triangleright)$	$P_0 \cup (P_0\triangleleft)$

In all cases, $x/I \leq \tau_i(x)$ follows directly from the definition of I. Moreover, (6) follows from (7).

Henceforth working in C, suppose that $y \in A'_i$, $t \in I$, and $x \leq y \vee t$. We will show that $\tau_i(x) \leq y \vee e$ for some $e \in I$. Every element of $A'_i \setminus \{1\}$ is a nonempty finite meet of elements of the form $z \vee w_i$ where $z \in A'_2 \setminus \{1\}$ and w_i is $\bigvee_{n\in R_i} b^n_i \vee \bigvee_{n\in S_i} -b^n_i$ where R_i and S_i are each (possibly empty) finite subsets of ω and $R_i \perp S_i$. (Our convention is that $\bigvee \varnothing = 0$.) Moreover, by (7), $\tau_i(x) = 1$ if and only if $h = -h_0 \wedge -h_1$ or $P_{1-i} = \varnothing$. Therefore, it is enough to assume that $y = z \vee w_i$ where z and w_i are as above and prove that $h \leq h_0 \vee h_1$, that $\tau_i(x) \leq y \vee e$ for some $e \in I$, and that $P_{1-i} \neq \varnothing$.

By assumption, we have $b_{1-i} \wedge h \leq z \vee w_i \vee t$; by independence of (A'_2, B_0, B_1, H), we have $b_{1-i} \wedge h \leq w_i \vee t$, which implies $b_{1-i} \wedge h \leq w_i \vee (h_0 \vee h_1)$; by independence of (B_0, B_1, H), we have $h \leq h_0 \vee h_1$. Choose $m < \omega$ and $e = \bigvee_{\substack{j<2 \\ n<m}} e_i^n$ such that $m \supseteq S_i$ and $e \geq t$. Let $g_{0,n}(0) = b_0^n$, $g_{1,n}(0) = b_0^{n+1}$, $g_{j,n}(1) = b_1^n$, and $g_{j,n}(2) = h_j$ for all $(j, n) \in 2 \times \omega$. Then

$$b_{1-i} \wedge h \leq w_i \vee e = w_i \vee \bigvee_{\substack{j<2 \\ n<m}} \bigwedge_{k<3} g_{j,n}(k) = w_i \vee \bigwedge_{f\colon 2\times m \to 3} \bigvee_{\substack{j<2 \\ n<m}} g_{j,n}(f(j,n)).$$

Thus, $b_{1-i} \wedge h \leq w_i \vee e$ if and only if, for all $f\colon 2 \times m \to 3$,

$$b_{1-i} \wedge h \leq w_i \vee \bigvee_{\substack{j<2 \\ n<m}} g_{j,n}(f(j,n)). \tag{8}$$

Given any $f\colon 2 \times m \to 3$, let, for each $(j,k) \in 2 \times 3$, $E_{j,k} = \{n : f(j,n) = k\}$; let $F_{1,0} = E_{1,0}\triangleright$ and $F_{1,1} = E_{1,1}$. Independence of (B_0, B_1, H), $(b_0^n)_{n<\omega}$, and $(b_1^n)_{n<\omega}$ imply that (8) is equivalent to the disjunction of

(X1) $P_{1-i} \not\subseteq E_{0,1-i} \cup F_{1,1-i}$;
(X2) $S_i \not\subseteq E_{0,i} \cup F_{1,i}$;
(X3) $h \leq h_0$ and $E_{0,2} \neq \varnothing$;
(X4) $h \leq h_1$ and $E_{1,2} \neq \varnothing$.

Similarly, we have $\tau_i(x) \leq w_i \vee e$ if and only if, for all $f\colon 2 \times m \to 3$, we have $S_i \not\subseteq E_{0,i} \cup F_{1,i}$ or $T_i \not\subseteq S_i$. By (7), $T_i \not\subseteq S_i$ if and only if $P_{1-i} \not\subseteq U_i$ where U_i is as in (9) below.

(9)

h	U_0	U_1
$h_0 \wedge -h_1$	S_0	S_1
$-h_0 \wedge h_1$	$S_0\triangleleft$	$S_1\triangleright$
$h_0 \wedge h_1$	$S_0 \cup (S_0\triangleleft)$	$S_1 \cup (S_1\triangleright)$

By choosing f according to (10) below, we ensure that (X2), (X3), and (X4) fail, and, therefore, that (X1) holds.

(10)

h	i	$E_{0,2}$	$E_{1,2}$	$E_{0,1-i}$	$E_{1,1-i}$	$m \setminus F_{1,i}$	$F_{1,1-i}$
$h_0 \wedge -h_1$	0	$\varnothing$	m	S_0	$\varnothing$	$\varnothing$	$\varnothing$
$h_0 \wedge -h_1$	1	$\varnothing$	m	S_1	$\varnothing$	$\varnothing$	$\varnothing$
$-h_0 \wedge h_1$	0	m	$\varnothing$	$\varnothing$	$S_0\triangleleft$	$S_0 \cup \{0\}$	$S_0\triangleleft$
$-h_0 \wedge h_1$	1	m	$\varnothing$	$\varnothing$	S_1	S_1	$S_1\triangleright$
$h_0 \wedge h_1$	0	$\varnothing$	$\varnothing$	S_0	$S_0\triangleleft$	$S_0 \cup \{0\}$	$S_0\triangleleft$
$h_0 \wedge h_1$	1	$\varnothing$	$\varnothing$	S_1	S_1	S_1	$S_1\triangleright$

Comparing (9) with (10), we see that $E_{0,1-i} \cup F_{1,1-i} = U_i$ in all cases. Therefore, $P_{1-i} \not\subseteq U_i$. Thus, $\tau_i(x) \leq y \vee e$ and $P_{1-i} \neq \varnothing$. □

Choose $A_\alpha = D_\alpha \oplus \mathrm{Fr}_\omega$ such that $A_\alpha \Subset M_\alpha$, that $\oplus_0 = \mathrm{id}_{D_\alpha}$, and that $A_\alpha \setminus D_\alpha$ is disjoint from $\bigcup_{\beta<\alpha} M_\beta$. Thus, (1) is preserved. By construction, $A_\alpha \cap \bigcup_{\beta<\alpha} M_\beta = \bigcup_{i<2} A'_{\alpha,i}$, so (2) is preserved. By Claim 5.4, (5) is preserved. By Sirota's Lemma, since $A'_{\alpha,i} \leq_{\mathrm{rc}} D_\alpha \leq_{\mathrm{rc}} A_\alpha$ for all $i < 2$ and D_α splits perfectly in A_α, (4) is preserved. Thus, our construction of Ω is complete and Ω has the FN.

Choose a long ω_1-approximation sequence $(N_\alpha)_{\alpha<\omega_2}$ with $N_\alpha \prec H(\aleph_3)$ for all $\alpha < \omega_2$ and $A, M \in N_0$ (which implies $\Omega \in N_0$). We will show that $\Omega[N]$ is not locally finite. Let $\delta = \omega_1 + 1$; let $\beta = \omega_2 \cap N_{\delta,0}$, which is in $\omega_2 \cap N_{\delta,1}$ and has cofinality ω_1 by Lemma 2.5; let $[\beta, \alpha) = [\beta, \beta + \omega_1) \cap N_{\delta,1}$. Note that $\alpha \in N_\delta$ and $\lfloor \alpha \rfloor_1 = \beta$.

Claim 5.7. $\beta \cap N_{\delta,1} = \beta \cap M_{\alpha,1}$.

Proof. By Lemma 2.3, $\beta \subseteq \bigcup_{\theta<\omega_1} M_{\beta+\theta}$. Since $\beta, M \in N_{\delta,1} \prec H(\aleph_3)$, we have

$$\beta \cap N_{\delta,1} = \beta \cap \bigcup\{M_\theta : \theta \in [\beta, \beta + \omega_1) \cap N_{\delta,1}\} = \beta \cap \bigcup_{\beta \le \theta < \alpha} M_\theta = \beta \cap M_{\alpha,1}. \quad \square$$

We will show that $\{h_{\alpha,0}, h_{\alpha,1}, b^0_{\alpha,0}\}$ generates an infinite subalgebra of $\Omega[N]$. It suffices to show that

- $\rho(b^m_{\alpha,i} \wedge h_{\alpha,i}, N) = \delta$ for all $i < 2$ and $m < \omega$,
- $\pi^0_+(b^m_{\alpha,1} \wedge h_{\alpha,1}, N) = -b^{m+1}_{\alpha,0}$ for all $m < \omega$, and
- $\pi^1_+(b^m_{\alpha,0} \wedge h_{\alpha,0}, N) = -b^m_{\alpha,1}$ for all $m < \omega$.

First, $\rho(b^m_{\alpha,i} \wedge h_{\alpha,i}, M) = \alpha$ by construction. By definition of α, we have $\alpha \in N_\delta \setminus N_{\delta,1}$; since $\beta \le \alpha$, we also have $\alpha \notin N_{\delta,0}$. Since $M, \rho(\bullet, M) \in N_0$ and M_α is countable, we conclude that $b^m_{\alpha,i} \wedge h_{\alpha,i} \in N_\delta \setminus \bigcup_{j<2} N_{\delta,j}$. Hence, $\rho(b^m_{\alpha,i} \wedge h_{\alpha,i}, N) = \delta$.

Second, we have, by Corollary 3.1 and Claim 5.6,

$$\pi^0_+(b^m_{\alpha,1} \wedge h_{\alpha,1}, N) = \bigwedge_{j<2} \pi^{N_{\delta,0}}_+(\pi^j_+(b^m_{\alpha,1} \wedge h_{\alpha,1}, M)) = \pi^{N_{\delta,0}}_+(-b^{m+1}_{\alpha,0}) \wedge \pi^{N_{\delta,0}}_+(b^m_{\alpha,1}).$$

We have $\pi^{N_{\delta,0}}_+(-b^{m+1}_{\alpha,0}) = -b^{m+1}_{\alpha,0}$ because $-b^{m+1}_{\alpha,0} \in N_{\delta,0}$ because $\rho(-b^{m+1}_{\alpha,0}, M) \in I_0(\alpha) = \beta$. We have $\pi^{N_{\delta,0}}_+(b^m_{\alpha,1}) = \pi^{A'_{\alpha,2}}_+(b^m_{\alpha,1}) = 1$ by Lemma 3.1 because $N_{\delta,0} \cap \Omega = A_{\alpha,0} \perp\!\!\!\perp A'_{\alpha,1}$ by (5) and Proposition 3.1, and because $A_{\alpha,0} \cap A'_{\alpha,1} = A'_{\alpha,2} \le_{\mathrm{rc}} A'_{\alpha,1}$. Thus, $\pi^0_+(b^m_{\alpha,1} \wedge h_{\alpha,1}, N) = -b^{m+1}_{\alpha,0}$.

Third, we have, by Corollary 3.1 and Claim 5.6,

$$\pi^1_+(b^m_{\alpha,0} \wedge h_{\alpha,0}, N) = \bigwedge_{j<2} \pi^{N_{\delta,1}}_+(\pi^j_+(b^m_{\alpha,0} \wedge h_{\alpha,0}, M)) = \pi^{N_{\delta,1}}_+(b^m_{\alpha,0}) \wedge \pi^{N_{\delta,1}}_+(-b^m_{\alpha,1}).$$

We have $\pi^{N_{\delta,1}}_+(-b^m_{\alpha,1}) = -b^m_{\alpha,1}$ because $\rho(-b^m_{\alpha,1}, M) \in I_1(\alpha) = [\beta, \alpha) \subseteq N_{\delta,1}$. We have $\pi^{N_{\delta,1}}_+(b^m_{\alpha,0}) = \pi^{A'_0 \cap N_{\delta,1}}_+(b^m_{\alpha,0})$ by Lemma 3.1 because $A'_{\alpha,0} \perp\!\!\!\perp (\Omega \cap N_{\delta,1})$ by (5) and Proposition 3.1, and because, arguing as in the proof of Claim 5.2, $A'_{\alpha,0} \cap N_{\delta,1} \le_{\mathrm{rc}} A'_{\alpha,0}$. Because $A'_{\alpha,0} \cap N_{\delta,1} = A'_{\alpha,2}$ by Claim 5.7, we also have $\pi^{A'_0 \cap N_{\delta,1}}_+(b^m_{\alpha,0}) = \pi^{A'_{\alpha,2}}_+(b^m_{\alpha,0}) = 1$. Thus, $\pi^1_+(b^m_{\alpha,0} \wedge h_{\alpha,0}, N) = -b^m_{\alpha,1}$.

Thus, Ω has the FN but not the SFN.

We briefly remark that the interaction between π^0_+ and π^1_+ is essential to the above construction. Given boolean algebras $K \le_{\mathrm{rc}} L$, it is not hard to check, using Lemma 3.5, that $(L, \wedge_L, \vee_L, -_L, 0_L, 1_L, \pi^K_+, \pi^K_-)$ is locally finite. This lemma can also be used to re-prove the implication from FN to SFN for boolean algebras of size at most $\aleph_1$, without using, as Heindorf and Shapiro do, the implication from FN to projectivity for boolean algebras of size at most $\aleph_1$.

Indeed, given a boolean algebra A of size at most $\aleph_1$ with the FN, let $(M_\alpha)_{\alpha<\omega_1}$ be a long ω_1-approximation sequence with $A \in M_0$, let $\mathcal{F}_1$ be a chain of finite subalgebras of $A \cap M_0$ with union $A \cap M_0$, and then inductively assume that $1 \le$

$\alpha < \omega_1$ and $\mathcal{F}_\alpha$ is a pairwise commuting cofinal family of finite subalgebras of $A \cap M_{\alpha,0}$. Let $A \cap M_\alpha = \{a_n : n < \omega\}$ and set $C_0 = \{0, 1\}$. Given $n < \omega$ and a finite $C_n \leq A \cap M_\alpha$, let A_n be the subalgebra of A generated by $C_n \cup \{a_n\}$; choose $B_n \in \mathcal{F}_\alpha$ containing $\pi_+^{M_{\alpha,0}}[A_n]$; let C_{n+1} be the subalgebra of A generated by $A_n \cup B_n$. By Lemma 3.5, $\pi_+^{M_{\alpha,0}}[C_{n+1}] = B_n$; hence, for all $D \in \mathcal{F}_\alpha$, $\pi_+^D[C_{n+1}] = (\pi_+^D \circ \pi_+^{M_{\alpha,0}})[C_{n+1}] = \pi_+^D[B_n]$, which implies $C_{n+1} \downarrow D$ by Lemma 3.2. Thus, $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{C_n : n < \omega\}$ is a pairwise commuting cofinal family of finite subalgebras of $A \cap M_{\alpha+1,0}$.

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