Between Tukey equivalence and Boolean automorphism

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Outline

- Motivating pin equivalence from Tukey equivalence
- A representation theorem
- Separating pin equivalence from Tukey equivalence
- Separating pin equivalence from automorphism equivalence
- Between pin equivalence and automorphism equivalence

Tukey equivalence in terms of joint embeddings

Definition

- A poset is **directed** if every finite set has an upper bound.
- Subset C of poset P is **cofinal** if every $p \in P$ has an upper bound in C.
- Two directed posets P, Q are **Tukey equivalent** if there is a poset D with cofinal subsets P', Q' order-isomorphic to P, Q.

Claim

Tukey equivalence is transitive.

Proof.

Suppose $P \equiv_T Q \equiv_T R$ is witnessed by P', Q' cofinal in D and Q'', R'' cofinal in E. Then let $F = (D \cup E)/\sim$ where $\sim$ identifies Q' and Q'' and $\leq_F$ is the transitive closure of $\leq_D \cup \leq_E \cup \sim$. $\square$

Directed posets may as well be Boolean ideals.

Notation

- $A \downarrow x = \{a \in A \mid a \leq x\}$
- $A \uparrow x = \{a \in A \mid a \geq x\}$
- $A \downarrow B = \bigcup_{x \in B} A \downarrow x$
- $A \uparrow B = \bigcup_{x \in B} A \uparrow x$

Claim

For any set $\mathcal{C}$ of directed posets, there a Boolean algebra A such that every $P \in \mathcal{C}$ is Tukey equivalent to an ideal of A.

Proof.

Assume $\mathcal{C}$ is pairwise disjoint. Let A be the Boolean algebra generated by set $\bigcup \mathcal{C}$ and relations $x \wedge y = x$ for $x \leq_P y$ for $P \in \mathcal{C}$. Each P is cofinal in the ideal $A \downarrow P$. $\square$

Tukey equivalence in terms of poset automorphism

Claim

Two ideals I, J of a Boolean algebra A are Tukey equivalent iff there is a poset $(P, \leq_P)$ extending $(A, \leq_A)$ and an order automorphism h of P mapping $P \downarrow I$ onto $P \downarrow J$.

Proof (sketch).

Suppose I, J have copies cofinal in poset C. Extend A to its Boolean completion B. Let P be B with $I \setminus J$, $J \setminus I$ replaced by copies D, E of C. Let $f\colon D \cong E$ and $g = f \cup f^{-1}$. Extend g to $h\colon P \cong P$ as follows.

$$g(x) = \left[x - \left[\bigvee((I \cup J) \downarrow x)\right]\right] \vee \left[\bigvee f((I \cup J) \downarrow x)\right] \text{ for } x \notin I \cup J;$$

$$g(x) = \left[\bigwedge f((I \setminus J) \uparrow x)\right] \wedge \left[\bigwedge f((J \setminus I) \uparrow x)\right] \text{ for } x \in I \cap J. \qquad \square$$

Pin equivalence

Claim (again)

Two ideals I, J of a Boolean algebra A are Tukey equivalent iff there is a poset $(P, \leq_P)$ extending $(A, \leq_A)$ and an order automorphism h of P mapping $P \downarrow I$ onto $P \downarrow J$.

Definition

Two ideals I, J of a Boolean algebra A are **pin equivalent** iff there is a Boolean algebra B extending A and a Boolean automorphism h of B mapping $B \downarrow I$ onto $B \downarrow J$.

By the above claim, pin equivalence implies Tukey equivalence. But we can also show this directly. If B, h witness $I \equiv_{\text{pin}} J$, then J is cofinal in $B \downarrow J$ and I has a copy $h(I)$ cofinal in $B \downarrow J$.

Pin equivalence in topology

In topology, we are particularly interested in neighborhood filters of points or, equivalently, the ideals dual to these filters.

Definition

Call two points a, b in a compact Hausdorff space X **pin equivalent** if there exist:

- a compact space Y,
- a continuous surjection $f\colon Y \to X$ invertible at a and b,
- and a homeomorphism $g\colon Y \to Y$ with $g(f^{-1}(a)) = f^{-1}(b)$.

By Stone duality, when X is zero-dimensional, a and b are pin equivalent iff the ideals I_a, I_b of clopen subsets of $X \setminus \{a\}, X \setminus \{b\}$ are pin equivalent ideals of the clopen algebra of X.

A representation theorem

Theorem

Points a, b in a compact Hausdorff space X are pin equivalent iff there is a closed symmetric binary relation $R \subset X^2$ with domain X such that $aRx \Leftrightarrow x = b$ and $bRx \Leftrightarrow x = a$.

(The above representation theorem is not so easy to express in terms of Boolean algebras.)

Corollary

Pin equivalence is transitive.

Proof (sketch).

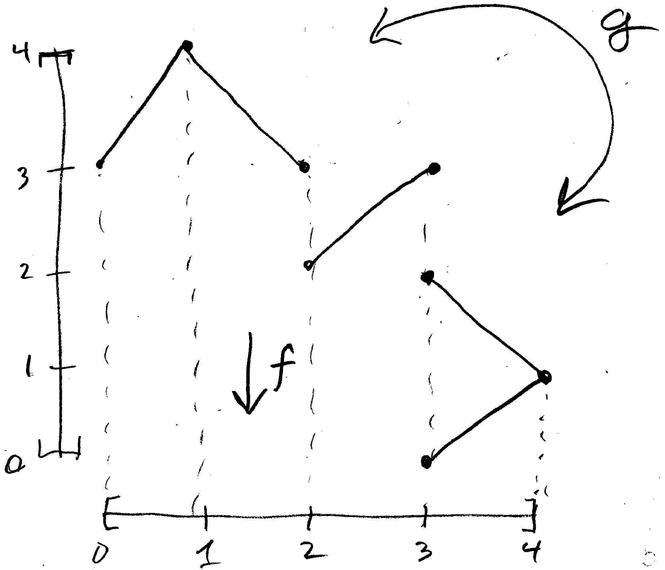
If a, b, c are distinct, $R_1 \subset X^2$ witnesses $a \equiv_{\text{pin}} b$, and $R_2 \subset X^2$ witnesses $b \equiv_{\text{pin}} c$, then $(R_1' \circ R_2') \cup (R_2' \circ R_1') \cup (\text{small nhbd. of } b)^2$ witnesses $a \equiv_{\text{pin}} c$ where $R_1' = R_1 \cap (X - (\text{small nhbd. of } c))^2$ and $R_2' = R_2 \cap (X - (\text{small nhbd. of } a))^2$ $\square$

Corollary

In any first countable compact Hausdorff space, all points are pin equivalent.

Example of the representation theorem

Let's see why 1 and 4 are pin equivalent in $X = [0, 4]$.



Pin equivalence strictly implies Tukey equivalence, part I

Topologically speaking, weak P-points are only pin equivalent to other weak P-points. Algebraically speaking:

Definition

An ideal I of a Boolean algebra A is **weak P-ideal** if, for any countable covering of I by ideals K_n of A for $n < \omega$, some K_m already contains I.

Claim

If I is a weak P-ideal of Boolean algebra A and $I \equiv_{pin} J$, then J is also a weak P-ideal of A.

Proof.

In a Boolean extension B, let $h\colon B \cong B$ map $B \downarrow I$ onto $B \downarrow J$. Suppose $J \subset \bigcup_{n<\omega} K_n$. Then $I \subset \bigcup_{n<\omega}(A \downarrow h^{-1}(K_n))$. Then $I \subset A \downarrow h^{-1}(K_m)$ for some m. Then $J \subset K_m$. □

Pin equivalence strictly implies Tukey equivalence, part II

Definition

- An ideal I is a P-**ideal** if every countable subset of I has an upper bound in I.
- An ideal I is κ-**OK** if for every $a\colon \omega \to I$ there exists $b\colon \kappa \to I$ such that for all $n < \omega$ and all $\xi_1 < \xi_2 < \ldots < \xi_n < \kappa$, we have
$$b(\xi_1) \vee \cdots \vee b(\xi_n) \geq a_n.$$

Theorem (Kunen, 1978)

$\mathcal{P}(\omega)/\mathrm{Fin}$ *contains a maximal ideal that is* $\mathfrak{c}$*-OK but is not a* P*-ideal.*

Pin equivalence strictly implies Tukey equivalence, part III

Theorem (Kunen, 1978)

$\mathcal{P}(\omega)/\mathrm{Fin}$ contains a maximal ideal that is $\mathfrak{c}$-OK but is not a P-ideal.

Corollary

There are weak P-ideals Tukey equivalent to ideals that are not weak P. In particular, Tukey equivalence does not imply pin equivalence.

Proof.

- $\mathfrak{c}$-OK implies ω_1-OK implies weak P.
- All $\mathfrak{c}$-OK non-P ideals of $\mathcal{P}(\omega)/\mathrm{Fin}$ are **Tukey-maximal**, that is, Tukey equivalent to $[\mathfrak{c}]^{<\omega}$.
- Fubini squares of Tukey-maximal ideals of $\mathcal{P}(\omega)/\mathrm{Fin}$ are Tukey-maximal.
- Fubini squares of non-principal ideals of $\mathcal{P}(\omega)/\mathrm{Fin}$ are not weak P.

Generalizing from $\mathcal{P}(\omega)/\mathrm{Fin}$

Definition

A Boolean algebra has the **countable separation property (CSP)** if, every two countably generated ideals I, J with $I \cap J = \{0\}$, extend to principal ideals I', J' with $I' \cap J' = \{0\}$.

(The compact Hausdorff spaces with the Stone dual of the CSP are called F-spaces.)

$\mathcal{P}(\omega)/\mathrm{Fin}$ has the CSP. We have seen that $\mathcal{P}(\omega)/\mathrm{Fin}$ has pin inequivalent maximal ideals. I can generalize this to all CSP Boolean algebras, assuming the existence of Rudin-Keisler incomparable selective ultrafilters on ω. In particular:

Theorem

Assume CH. Every CSP Boolean algebra has pin inequivalent maximal ideals.

Question: Is the above theorem is true in ZFC?

Homeomorphism/automorphism types

Definition

- Points a, b in a topological space X have the same **homeomorphism type** if there is a homeomorphism $h\colon X \cong X$ such that $h(a) = b$.
- Dually, ideals I, J of a Boolean algebra A have the same **automorphism type** if there is a Boolean automorphism $h\colon A \cong A$ such that $h(I) = J$.

Theorem (Kunen)

Every CSP Boolean algebra has maximal ideals with different automorphism types (without any assumptions beyond ZFC).

Pin equivalence: much coarser than homeomorphism type

Example

There are points in the compact Hausdorff space $2^\omega \times 2^{\omega_1}_{\mathsf{lex}}$ with different π-characters, yet all these points are pin equivalent.

Theorem

If X is compact Hausdorff, then all points in $X \times 2^{\chi(X)}$ are pin equivalent.

Corollary

$(\beta\mathbb{N} \setminus \mathbb{N}) \times 2^{\mathfrak{c}}$ *has pin equivalent points with different homeomorphism types.*

Proof.

Kunen has has shown that points with different homeomorphism types exist in any compact Hausdorff product of one or more F-spaces and zero or more first countable spaces. $\square$

Between pin equivalence and homeomorphism type

Definition

- A subalgebra A of a Boolean algebra B is **relatively complete (rc)** if, for every principal ideal I of B, the ideal $A \cap I$ of A is also principal.
- Ideals I, J of a Boolean algebra A are **rc-pin symmetric** if there is an rc Boolean extension B of A and an automorphism $h\colon B \cong B$ such that $h(B \downarrow I) = B \downarrow J$.

The Stone dual of a relative complete Boolean embedding is an open continuous surjection. So, let **open pin symmetry** denote the Stone dual of rc-pin symmetry.

Remark

Unlike pin equivalence, open pin symmetry preserves π-character.

Open problems

Question

Does every Boolean algebra A have a Boolean extension A' such every two maximal ideals of A' have the same automorphism type?

Kunen formulated this question topologically in 1990. Van Douwen formulated a special case (which is still open) c. 1970.

If we replace "have the same automorphism type" with "are pin equivalent," then the answer is yes: extend A to a coproduct of A with a sufficiently large free Boolean algebra. This is the Stone dual of my result about $X \times 2^{\chi(X)}$.

What if we replace "have the same automorphism type" with "are rc pin symmetric"?