# Pin homogeneity

David Milovich
http://dkmj.org

Texas A&M International University

March 17, 2018
52nd Spring Topology and Dynamical Systems Conference
Auburn University

# Compact homogeneous spaces

- (All spaces are assumed to be Hausdorff.)
- A space $X$ is **homogeneous** if, for all $p, q \in X$ there is a homeomorphism $h\colon X \to X$ with $h(p) = q$.
- Every compact space $X$ embeds into $([0,1]^\omega)^{w(X)}$, which is compact and homogeneous.
- But from another point of view "large" compact homogeneous spaces are hard to come by. No matter what model of ZFC we're in, here are some these things we don't know:
  - (W. Rudin) Is there an infinite compact homogeneous space with no nontrivial converging $\omega$-sequences?
  - (Van Douwen) Is there a compact homogeneous space with $\mathfrak{c}^+$-many disjoint open sets?
  - (Kunen) Is every compact space $X$ a continuous image of a compact homogeneous space?
    - For examples, what about $X = \omega_1 + 1$ or $X = \beta\mathbb{N} \setminus \mathbb{N}$?
  - (M.) Is there an infinite compact homogeneous space $X$ with a local base not Tukey equivalent to $[\chi(X)]^{<\omega}$?

# Some partial progress

We do know a few things. Perhaps most importantly, we know that infinite compact F-spaces are not homogeneous.

• (Frolik, M.E. Rudin) $\beta\mathbb{N} \setminus \mathbb{N}$ is not homogeneous because each $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$ is not a $\mathcal{U}$-limit of a discrete sequence.

• $\beta\mathbb{N} \setminus \mathbb{N}$ is an **F-space**, *i.e.*, if $U$ and $V$ are disjoint open $F_\sigma$ sets, then $U$ and $V$ have disjoint closures.

• (Kunen) If $X$ is compact and homogeneous then $X$ is not a product of one or more infinite F-spaces and zero or more first countable spaces.

• Kunen's proof uses his theorem that there exist Rudin-Keisler incomparable weak P-points in $\beta\mathbb{N} \setminus \mathbb{N}$.

# More partial progress

If $X$ is compact homogeneous then, in addition to Kunen's theorem, we have these restrictions:

- (De La Vega) $|X| \leq 2^{t(X)}$.
- (Van Mill) $|X| \leq 2^{\pi\chi(X)c(X)}$.
- (Ridderbos) $|X| \leq d(X)^{\pi\chi(X)}$.
- (M.) [GCH] Every local base in $X$ is Tukey-below $[\chi(X)]^{<c(X)}$.

There are also a few ways to cook up "interesting" compact homogeneous spaces $X$ (not including compact groups and metrizable spaces). Examples:

- (Maurice) Linearly ordered $X$ with $c(X) = \mathfrak{c}$
- (Dow, Pearl) $X = Y^\omega$ for any zero dimensional first countable compact $Y$
- (Van Mill) Resolutions: $X$ with $\pi(X) < \chi(X) < \mathfrak{p}$ (assuming $\omega_1 < \mathfrak{p}$)
- (M.) Amalgams: path connected $X$ with $c(X) \geq c(Y)$ for any compact homogenous $Y$

# A slight weakening of homogeneity

• Call a space $X$ **power homogeneous** if it has a homogeneous power.

• Finding "large" compact power homogeneneous spaces appears no easier than large finding compact homogeneous spaces.

• Many restrictions on homogeneous spaces also apply to power homogeneous spaces. For examples, if $X$ is power homogeneous and compact, then:

- (Kunen) $X$ is an not a product of one or more infinite F-spaces and zero or more first countable spaces.
- (Arhangel′skiĭ, Van Mill, Ridderbos) $|X| \leq 2^{t(X)}$.
- (Carlson, Ridderbos) $|X| \leq 2^{\pi\chi(X)c(X)}$
- (Ridderbos) $|X| \leq d(X)^{\pi\chi(X)}$.

# An extreme weakening of homogeneity

• Call a space **B-homogeneous** if it has a base whose elements are homeomorphic to each other.

• B-homogeneity is very different than homogeneity. In particular, every compact space $X$ is a continuous image of a B-homogeneous compact space. Proof:

- $X$ is a continuous image of $Y = X \times 2^{|X|}$ and all nonempty open subsets of $Y$ have equal cardinality.
- Let $D$ be $Y$ with the discrete topology.
- Extend $\mathrm{id}\colon D \to Y$ to $\beta\,\mathrm{id}\colon \beta D \to Y$.
- $\beta\,\mathrm{id}$ maps the subspace $u(D)$ of all uniform ultrafilters on $D$ onto $Y$.
- $u(D)$ is compact and B-homogeneous.

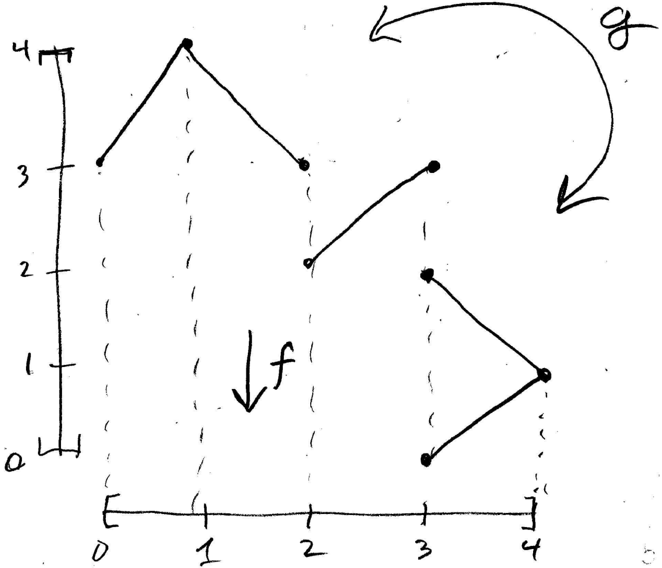
• Also note that every $u(D)$, including $u(\omega) = \beta\mathbb{N} \setminus \mathbb{N}$, **is B-homogeneous despite being a compact F-space.**

# Pin homogeneity: a slightly less extreme weakening

- Call two points $a, b$ in a compact space $X$ **pin equivalent** if there exist:
  - a compact space $Y$,
  - a continuous surjection $f\colon Y \to X$ invertible at $a$ and $b$,
  - and a homeomorphism $g\colon Y \to Y$ with $g(f^{-1}(a)) = f^{-1}(b)$.
- Two facts about pin equivalence:
  - Pin equivalence is local: $a, b \in X$ are pin equivalent in $X$ iff they are pin equivalent in some closed neighborhood of $\{a, b\}$.
  - Pin equivalence is transitive.
    - This is not obvious. The proof involves a fiber product.
- Call $X$ **pin homogeneous** if every two points in $X$ are pin equivalent.
- If we required $f$ to be invertible everywhere, then pin homogeneity would be homogeneity.

# Example: Closed intervals are pin homogeneous.

Let's see why 1 and 4 are pin equivalent in $X = [0, 4]$.



## $X^2$ witnesses every pin equivalence.

**Theorem**

*Points $a, b$ in a compact space $X$ are pin equivalent iff there exists $Y \subset X^2$ such that:*

- *$Y$ is symmetric and closed,*
- *the first coordinate projection $\pi \colon Y \to X$ is surjective,*
- *$\pi$ is invertible at $a$ and $b$,*
- *and $(a, b) \in Y$.*

I proved the above theorem by first characterizing pin equivalence in terms of Boolean algebras...

# Pin equivalence is a local Boolean property.

Call two filter bases $F, G$ in a Boolean algebra $A$ **pin equivalent** if there exist:

- an extension of $A$ to a Boolean algebra $A'$
- and an automorphism $h$ of $A'$ such that $h[F'] = G'$ where $F'$ and $G'$ are the filters of $A'$ generated by $F$ and $G$, respectively.

## Theorem

*Points $p, q$ in a compact space $X$ are pin equivalent iff they respectively have neighborhood bases $\mathcal{U}, \mathcal{V}$ such that:*

- $\overline{\bigcup \mathcal{U}}$ *and* $\overline{\bigcup \mathcal{V}}$ *are disjoint*
- *and $\mathcal{U}$ and $\mathcal{V}$ are pin equivalent in the Boolean subalgebra of $\mathcal{P}(X)$ generated by $\mathcal{U} \cup \mathcal{V}$.*

## Corollary

*Given $Y$ compact, $X = Y \times 2^{\chi(Y)}$* **is pin homogeneous.**

## Proof.

Each $p \in X$ has an independent local subbase of size $\chi(Y)$. $\square$

# Pin equivalence implies Tukey equivalence.

- Call two upwards directed posets $P, Q$ **Tukey equivalent** if there is an upwards directed poset $R$ with cofinal subsets $P', Q'$ isomorphic to $P, Q$, respectively.
- All the local bases at a given point in a space are Tukey equivalent to each other. (These local bases are ordered by $\supset$.)
- Call two points in a space **Tukey equivalent** if they have Tukey equivalent local bases.
- Pin equivalent points are Tukey equivalent. (Proof: Use the Boolean algebra characterization.)

# Pin homogeneity and F-spaces

### Theorem (CH)

*If $X$ is an infinite compact F-space, then it is not pin homogeneous.*

My proof is similar to that of Kunen's theorem. It uses RK-incomparable selective ultrafilters instead of merely RK-incomparable weak P-points.

Does ZFC already prove the above theorem? It does in the special case $X = \beta\mathbb{N} \setminus \mathbb{N}$:

### Theorem

$\beta\mathbb{N} \setminus \mathbb{N}$ *is not pin homogeneous.*

### Proof.

Pin equivalence preserves the property of being a weak $P$-point. □

# Open pin homogeneity: an intermediate weakening?

• Call two points $a, b$ in a compact space $X$ **open pin symmetric** if there exist:

- a compact space $Y$,
- an *open* continuous surjection $f\colon Y \to X$ that is invertible at $a$ and $b$,
- and a homeomorphism $g\colon Y \to Y$ such that $g(f^{-1}(a)) = f^{-1}(b)$.

• Open pin symmetry preserves, for example, $\pi$-character and the property of being the limit of an $\omega$-sequence.

• Call $X$ **open pin homogeneous** if every two points in $X$ are open pin symmetric.

• Is every compact space a continuous image of an open pin homogeneous compact space?

• What if we replace "open" with "quasi-open"? (Quasi-open maps map open sets to sets with nonempty interior.)