Future value of income/deposit stream (3-3).

$t=0$: A = account balance = 0

$t=\frac{1}{12}$: add PMT to A

Now $A = PMT$

$t=\frac{2}{12}$: ① Earn interest: multiply A by $1+\frac{r}{12}$

② add PMT to A

Now $A = PMT(1+\frac{r}{12}) + PMT$

$t=\frac{3}{12}$: ① Earn interest: multiply A by $1+\frac{r}{12}$

② add PMT to A

Now $A = PMT(1+\frac{r}{12})^2 + PMT(1+\frac{r}{12}) + PMT$

$t=\frac{4}{12}$: ① Earn interest & ② add PMT

Now: $A = PMT(1+\frac{r}{12})^3$

$+ PMT(1+\frac{r}{12})^2$

$+ PMT(1+\frac{r}{12})$

$+ PMT$

$\vdots$

$t=\frac{n}{12}$: Now:

$A = PMT(1-\frac{r}{12})^{n-1} + \cdots + PMT$

$\left(= \sum_{k=0}^{n-1} PMT(1-\frac{r}{12})^k \right)$

$t = \frac{4}{12}$: $A = PMT\left(1+\frac{r}{12}\right)^3 + PMT\left(1+\frac{r}{12}\right)^2$
$+ PMT\left(1+\frac{r}{12}\right) + PMT$

Let's write $i = r/12$.

$$A = PMT(1+i)^3 + PMT(1+i)^2 + PMT(1+i) + PMT$$

$$Ai = A((1+i)-1) = A(1+i) - A$$

$$Ai = \left[PMT(1+i)^4 + \cancel{PMT(1+i)^3} + \cancel{PMT(1+i)^2} + \cancel{PMT(1+i)}\right] - \left[\cancel{PMT(1+i)^3} + \cancel{PMT(1+i)^2} + \cancel{PMT(1+i)} + PMT\right]$$

$$Ai = PMT(1+i)^4 - PMT$$

$$Ai = PMT[(1+i)^4 - 1]$$

$$A = PMT[(1+i)^4 - 1]/i \quad \left(\text{at } t = \frac{4}{12}\right)$$

$$A = PMT\frac{(1+i)^n - 1}{i} \quad \text{at } t = \frac{n}{12}$$

Suppose you deposit \$100 into your monthly compounded $\underbrace{1.5\%\ \text{APR}}_{r}$ savings account, starting a month from now

$$t = \tfrac{0}{12}: \quad A = 0$$

$$t = \tfrac{1}{12}: \quad A = PMT = 100$$

$$t = \tfrac{2}{12}: \quad A = PMT\,\frac{(1+i)^2 - 1}{i} = 200.125$$

$$i = \frac{r}{12} = \frac{0.015}{12}$$

$$t = \tfrac{3}{12}: \quad A = 300.375\ldots$$

$$t = \tfrac{12}{12} = 1: \quad A = PMT\,\frac{(1+i)^{12} - 1}{i}$$

$$A = 1{,}208.28\ldots$$

$$A = PMT \cdot (((1+(r/12))\verb|^|12) - 1)/(r/12)$$

$$A \approx 12\,PMT\left(1 + \frac{12i}{2}\right) = 1{,}208$$

$$t = \tfrac{120}{12} = 10: \quad A = PMT\,\frac{(1+i)^{120} - 1}{i}$$

$$A = \underbrace{PMT}_{100} \cdot (((1+(\underbrace{r}_{0.015}/12))\verb|^|120 - 1)/(\underbrace{r}_{0.015}/12)$$

$$A = 12{,}938.03\ldots$$

$$A \approx 120\,PMT\left(1 + \frac{120i}{2}\right) = 12{,}900$$

works better when i is bigger or n is bigger

Another estimate:

$$A \approx \underbrace{120\,PMT(1+i)^{120/2}}_{12,934.00\ldots}$$

Try $PMT = 1,000$, $t = \frac{360}{12} = 30$, $r = 5\%$. (Home mortgage.) $\boxed{n = 360}$

$$A = \underbrace{1000\,\frac{\left(1+\frac{0.05}{12}\right)^{360} - 1}{0.05/12}}_{832,258.64} = PMT\,\frac{(1+i)^n - 1}{i}$$

Estimates $A \overset{?}{\approx} \underbrace{\underbrace{360}_{n}\,PMT\left(1+\frac{ni}{2}\right)}_{630,000}$ not good estimate

$$A \overset{?}{\approx} \underbrace{n\,PMT(1+i)^{n/2}}_{760,933.42}$$ not as bad

See also 3-3 for more examples.

HW IF you deposit 200 into a 1.7% APR monthly compounded savings account, ~~how much will~~ every month, then how much will be in the account 8 years from now?

How long will it take to reach an account balance of 50,000?

Repeat with $r = 2\%$.