Product rule & quotient rule (11-3)

Suppose you're selling lattes with a current price of \$2.50 per cup and current sales of 230 cups per day. Suppose you're raising the price at a rate of 0.01/day but sales are decreasing at a rate of 0.7 cups/day/day. What is the rate of change of revenue?

(\$/day)	(\$/cup)	(cups/day)
R = revenue	p = price	x = quantity

t = time (days)

Right now: $\left[\begin{array}{ll} p = 2.5 & x = 230 \\ \dfrac{dp}{dt} = 0.01 & \dfrac{dx}{dt} = -0.7 \end{array}\right]$

Always: $R = px$ (= \$575 /day right now)

What is $\frac{dR}{dt}$ right now?

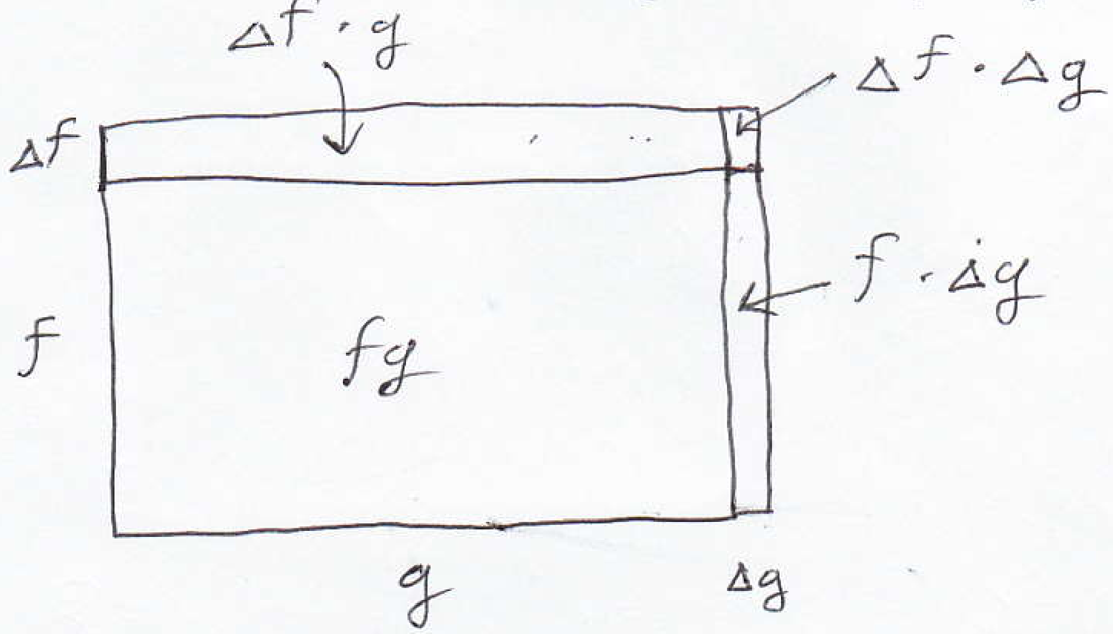
$$\frac{dR}{dt} = \frac{dp}{dt}x + p\frac{dx}{dt} = \underbrace{\overbrace{(0.01)(2.30)}^{2.3} + \overbrace{(2.5)(-.7)}^{-1.75}}_{.55\ (\$/\text{day})}$$

$$\frac{d(px)}{dt} = \frac{dp\cdot x}{dt} + \frac{p\cdot dx}{dt}$$

In general, $d(fg) = df\cdot g + f\cdot dg$

$$\Delta(fg) \approx \Delta f\cdot g + f\cdot\Delta g$$

(for Δf, Δg small)



$$\Delta(fg) = \Delta f\cdot g + \Delta f\cdot\Delta g + f\cdot\Delta g \approx \Delta f\cdot g + f\cdot\Delta g$$

What is slope of the line tangent to $y = x^2 e^{-x}$ at $x = 1$?

$dy/dx = ?$ when $x = 1$?

$$dy = d(\underbrace{x^2}_{f}\underbrace{e^{-x}}_{g}) = \underbrace{d(x^2)e^{-x}}_{df \cdot g} + \underbrace{x^2 d(e^{-x})}_{f \cdot dg}$$

$$d(x^n) = nx^{n-1}dx \Rightarrow d(x^2) = 2x^1 dx = 2x\,dx$$

$$d(e^x) = e^x dx \Rightarrow d(e^{-x}) = e^{-x} d(-x)$$

$$d(-x) = d(\underbrace{-1}_{\text{constant multiple}} \cdot x) = -1 \cdot dx = -dx$$

$$d(e^{-x}) = e^{-x}(-dx)$$

$$dy = (2x\,dx)e^{-x} + x^2(e^{-x}(-dx))$$

$$dy/dx = 2xe^{-x} + x^2 e^{-x}(-1)$$

At $x=1$: $2(1)e^{-1} + (1)^2 e^{-1}(-1)$

$$dy/dx = (2-1)e^{-1} = \boxed{e^{-1}}$$

If you want the tangent line equation, use that slope $m = e^{-1}$

in $y = y_0 + m(x - x_0)$

with $x_0 = 1$ & $y_0 = 1^2 e^{-1} = e^{-1}$:

Tangent line: $y = e^{-1} + e^{-1}(x-1)$

Quotiet rule: $\quad d\left(\frac{f}{g}\right) = \frac{df \cdot g - f \cdot dg}{g^2}$

Compare to

Product rule: $\quad d(fg) = df \cdot g + f \cdot dg$

Why? $\quad f = \left(\frac{f}{g}\right) \cdot g$

$$df = d(f) = d\left(\left(\frac{f}{g}\right) \cdot g\right) \underset{\uparrow}{=} d\left(\frac{f}{g}\right) g + \left(\frac{f}{g}\right) dg$$

Product rule

Solve for $d\left(\frac{f}{g}\right)$:

$$df - \left(\frac{f}{g}\right) dg = d\left(\frac{f}{g}\right) g$$

$$\frac{g}{g} \cdot \frac{df}{g} - \left(\frac{f}{g}\right) \frac{dg}{g} = d\left(\frac{f}{g}\right)$$

$$\frac{df \cdot g - f \cdot dg}{g^2} = d\left(\frac{f}{g}\right)$$

If a country's GDP is \$13 (trillion dollars) & population is 309 (million people) & GDP is increasing at .25 (trillion \$) per year & population is increasing at 3.2 (million people) per year, then what is the rate of change of GDP/person?

$Y = GDP = 13 \qquad \frac{dY}{dt} = .25$

$N = \text{population} = 309 \qquad \frac{dN}{dt} = 3.2$

$$\frac{d\left(\frac{Y}{N}\right)}{dt} = ? = \frac{\frac{dY \cdot N - Y \cdot dN}{N^2}}{dt}$$

$$= \frac{(dY/dt) \cdot N - Y(dN/dt)}{N^2}$$

$= 3.7 \cdot 10^{-4}$ \$$\underbrace{\text{trillions}}_{10^{12}}$ / ($\underbrace{\text{millions}}_{10^{6}}$ people) / year

$= \$370$/person per year

HW: (11-3) #11, 23, 85, 87
Due next class (Monday).