9/1 • Quotient rule } 11-3

• slopes

• rates of change } 10-4

Last time: $\boxed{d(fg) = (df)g + f(dg)}$

same $\begin{cases} (fg)' = f'g + fg' \\ \dfrac{d(fg)}{dx} = \dfrac{df}{dx}\, g + f\,\dfrac{dg}{dx} \end{cases}$

Product rule Quotient rule

$$d\left(\frac{f}{g}\right) = \frac{(df)\,g - f\,(dg)}{g^2}$$

$f = 12$ $g = 3$ (old)

$df = \Delta f = 0.1$ $dg = \Delta g = 0.2$ (change)

$f + \Delta f = 12.1$ $g + \Delta g = 3.2$ (new)

$f/g = 12/3 = 4$ (old)

$\Delta(f/g) = ?$ (change) $\dfrac{f+\Delta f}{g+\Delta g} = ?$ (new)

$$\Delta\left(\frac{f}{g}\right) \approx d\left(\frac{f}{g}\right) = \frac{(df)g - f(dg)}{g^2}$$

$$= \frac{(0.1)(3) - (12)(0.2)}{3^2} = \frac{-2.1}{9} = \frac{-0.7}{3}$$

$$= -0.23333\ldots$$

$$\frac{f+\Delta f}{g+\Delta g} = \frac{f}{g} + \Delta\left(\frac{f}{g}\right) \approx \frac{f}{g} + d\left(\frac{f}{g}\right)$$

$$\frac{f+\Delta f}{g+\Delta g} \approx \underbrace{4 + -0.23333\ldots}_{3.766666\ldots}$$

$$\frac{f+\Delta f}{g+\Delta g} = \frac{12.1}{3.2} = 3.78125$$

Derivative version:

$f(x),\ g(x)$

$$\left(\frac{f}{g}\right)' = \frac{d(f/g)}{dx} = \frac{\frac{(df)g - f(dg)}{g^2}}{dx}$$

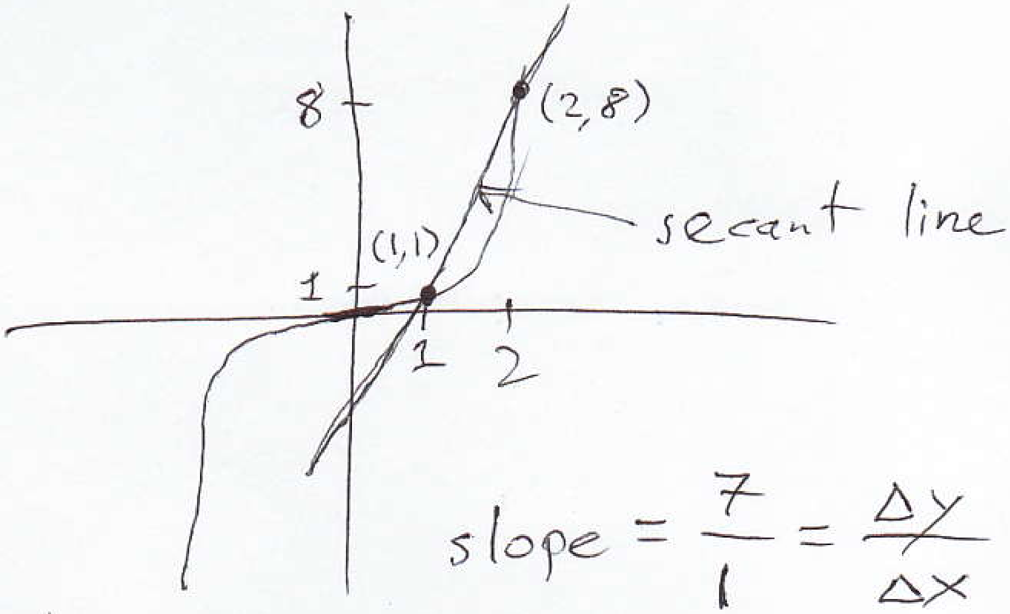
$$= \frac{(df/dx)g - f(dg/dx)}{g^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

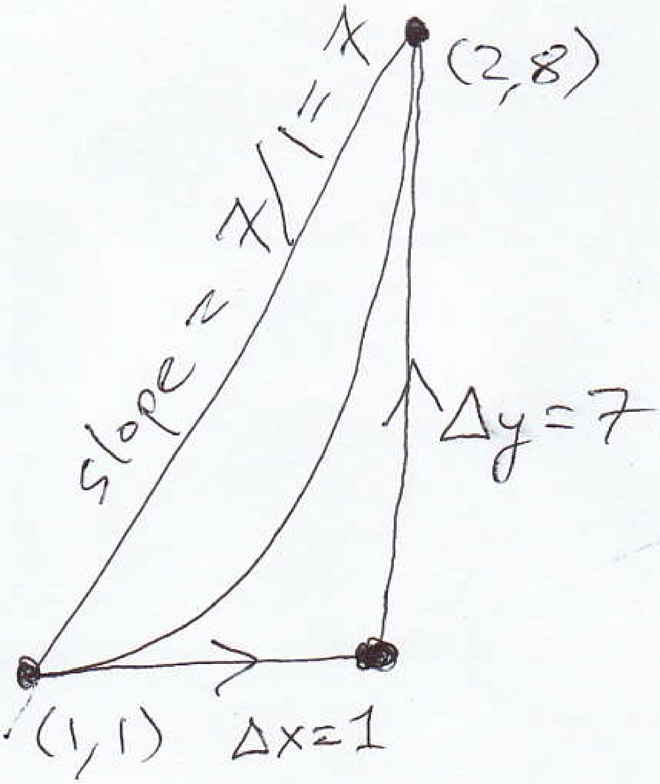
Compare to $f'g + fg' = (fg)'$

11-3 #22, 49, 52 : HW

$y = x^3$



$$\text{slope} = \frac{7}{1} = \frac{\Delta y}{\Delta x}$$



$\Delta y = 8 - 1 = 7$

$\Delta x = 2 - 1 = 1$

or:

$\Delta y = 1 - 8 = -7$

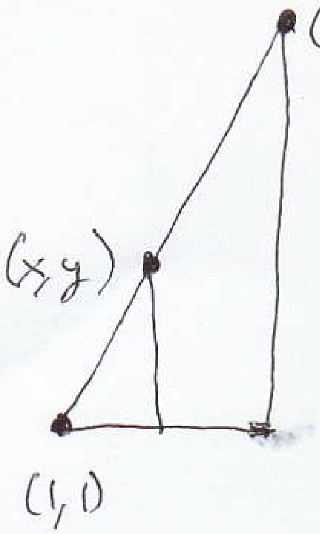
$\Delta x = 1 - 2 = -1$

$$\frac{\Delta y}{\Delta x} = \frac{-7}{-1} = \frac{7}{1}$$

secant line is $y = 7x - 6$

$\uparrow$ slope $\quad \uparrow$ y-intercept

My favorite method for secant line formula:



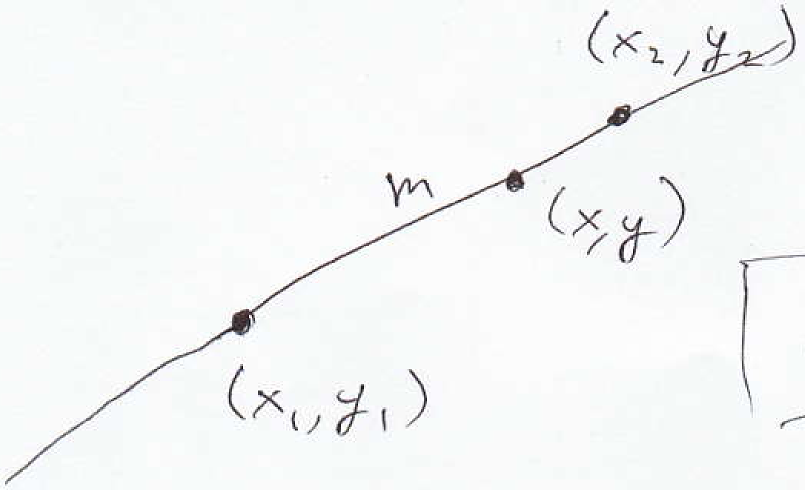
$$\frac{y-1}{x-1} = \text{slope} = \frac{8-1}{2-1} = 7$$

$$\frac{y-1}{x-1} = 7$$

$$y - 1 = 7(x-1)$$

same as

$$y = 7x - 6$$



$$\boxed{y - y_1 = m(x - x_1)}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$\frac{\Delta y}{\Delta x}$ = slope of secant line

when Δx is small, $dy \approx \Delta y$:

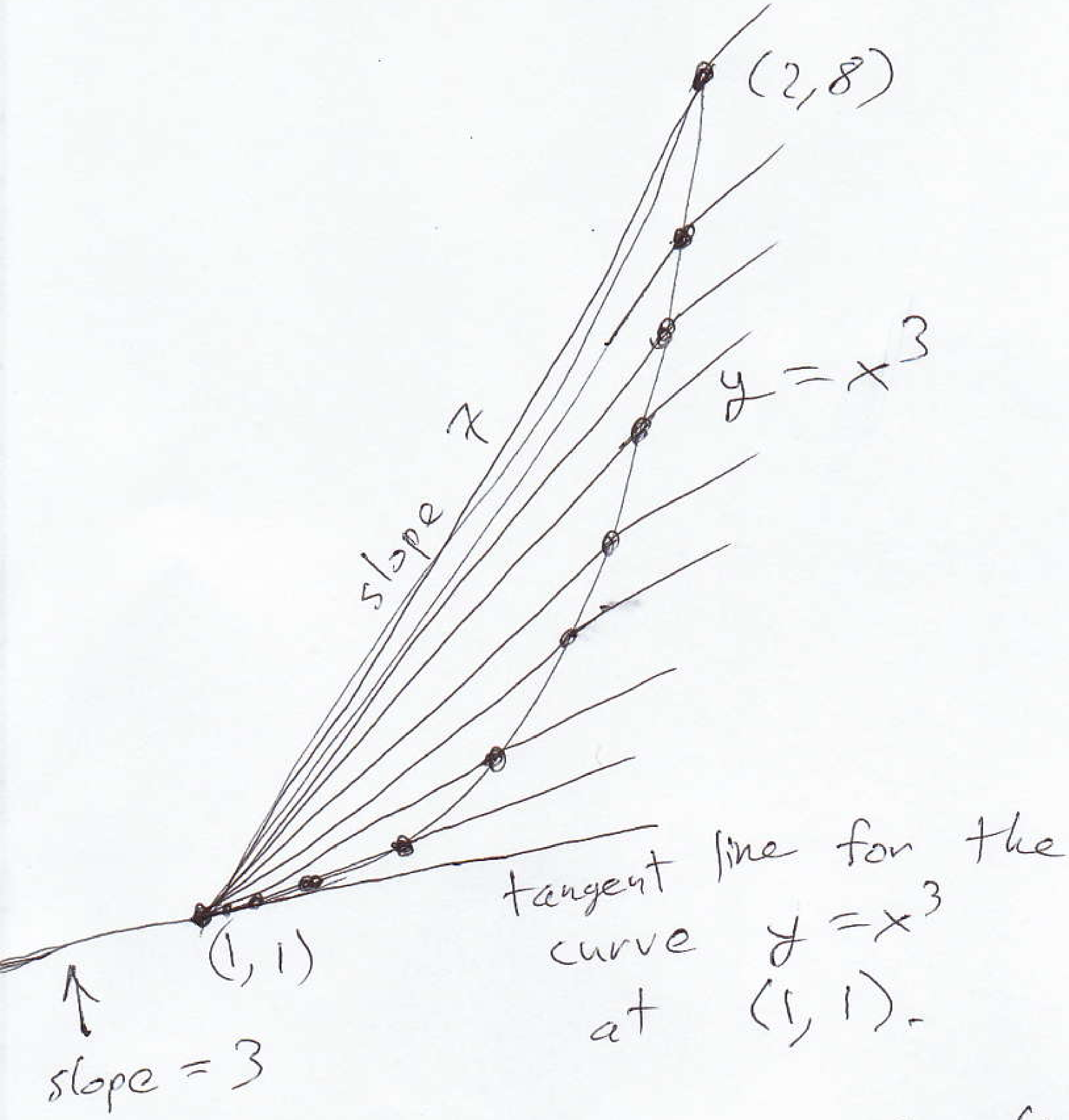
$$dy = \Delta y + (\text{small})\Delta x$$

$$dx = \Delta x \Rightarrow \frac{dy}{dx} = \frac{\Delta y + (\text{small})\Delta x}{\Delta x}$$

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} + \text{small}$$

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$$ ← slope of secant line

$\frac{dy}{dx}$ is exactly the slope of the tangent line



secant line from $(1,1)$ to $(1.1, \underbrace{1.1^3}_{1.331})$

$$\frac{\Delta y}{\Delta x} = \frac{1.331 - 1}{1.1 - 1} = \frac{.331}{.1} = 3.31$$

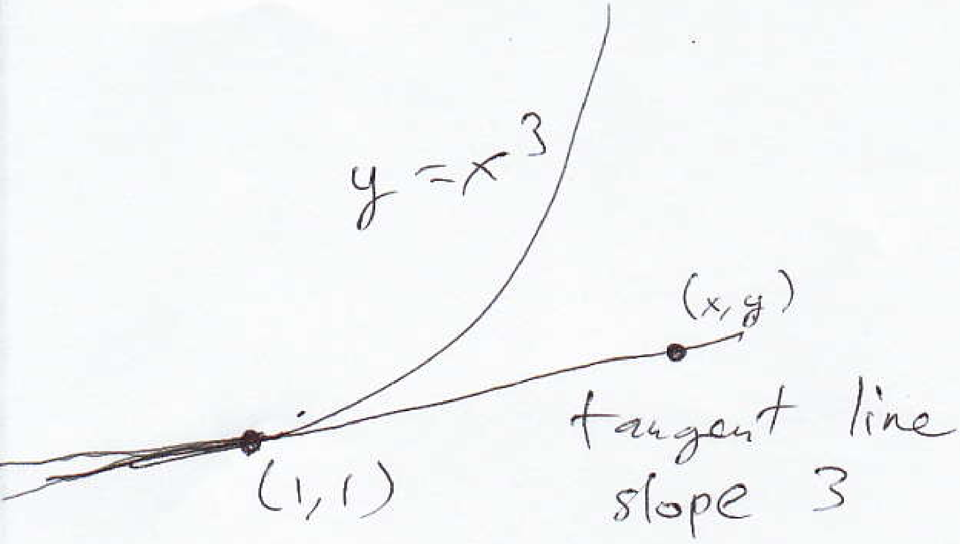
From $(1,1)$ to $(1.01, 1.01^3)$

$$\frac{\Delta y}{\Delta x} = \frac{1.01^3 - 1}{1.01 - 1} = \frac{0.03\ldots}{0.01} = 3.0301$$

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} = \frac{d(x^3)}{dx} = (x^3)' = 3x^2 = 3$$

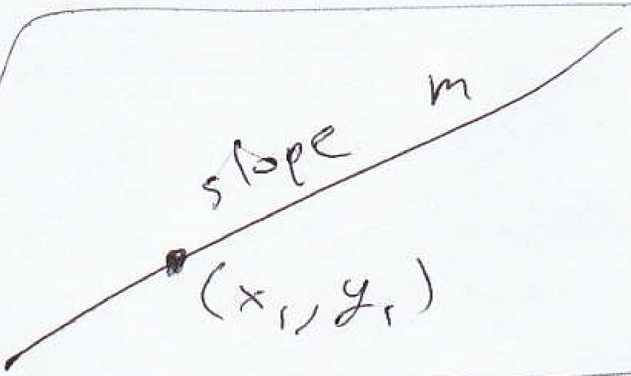
$x = 1$

slope of tangent line



$$\text{tangent line: } 3 = \frac{\Delta y}{\Delta x} = \frac{y-1}{x-1}$$

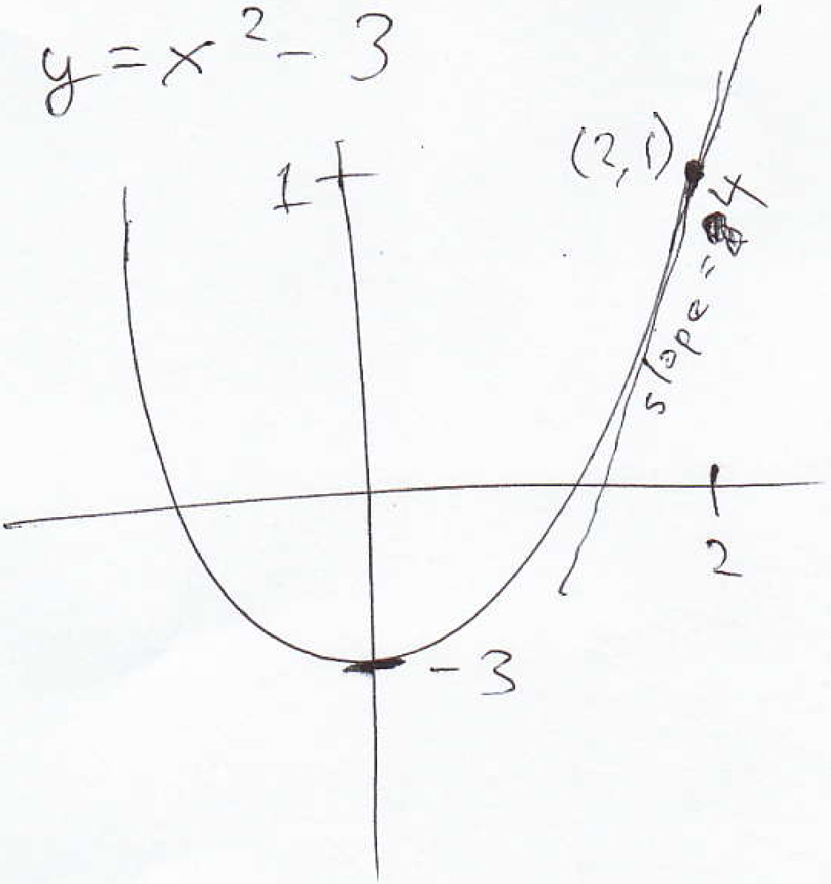
$$\text{equation: } 3(x-1) = y-1$$



$$y - y_1 = m(x - x_1)$$

HW: 10–4 #39

$y = x^2 - 3$



I pick

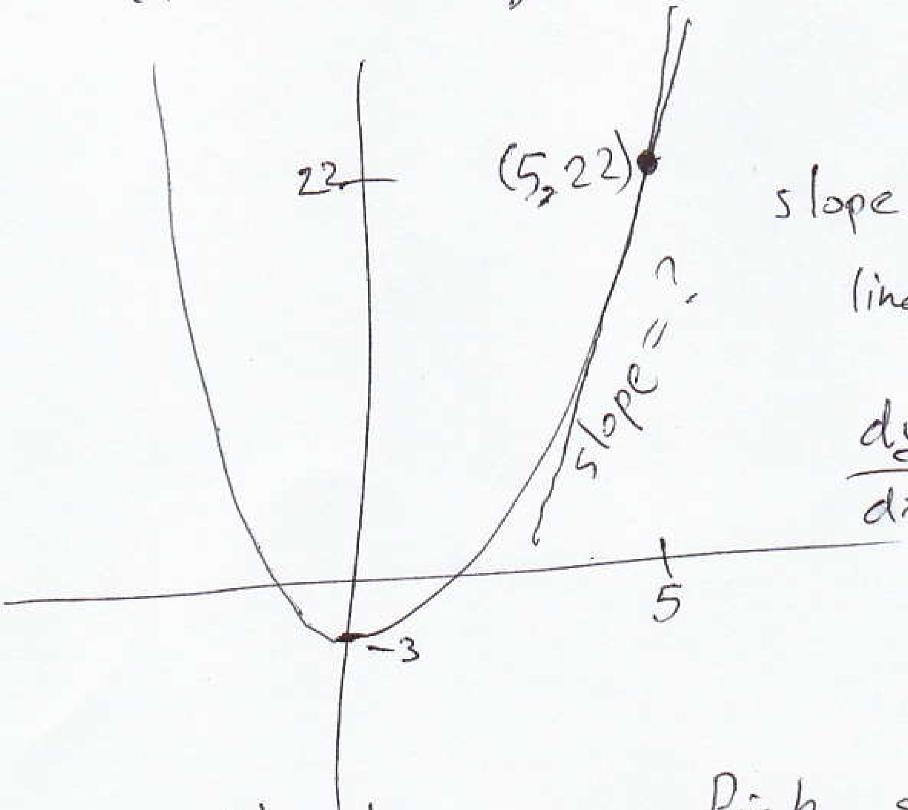
$$\frac{x = 2}{y = 2^2 - 3 = 1}$$

$$\frac{dy}{dx} = 2x$$

$$2(2) = 4$$

↑ slope of tangent line

$x = 5$ $\quad y = 5^2 - 3 = 22$



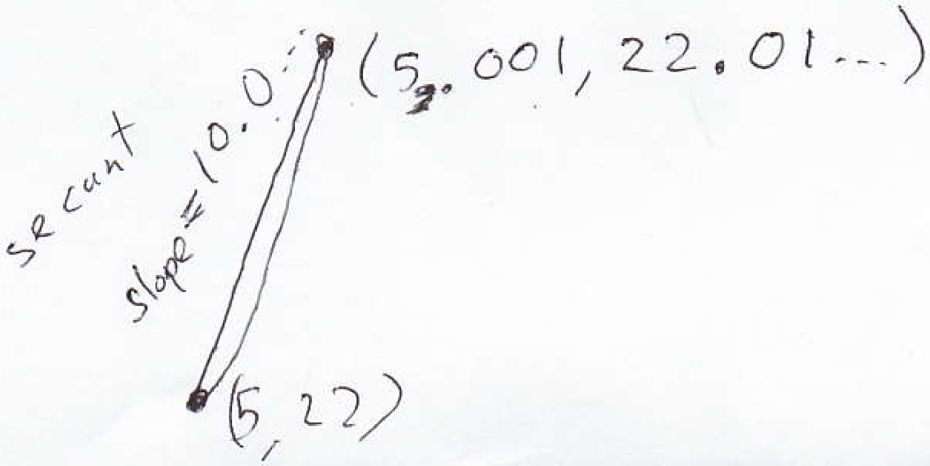
Easy way: slope of tangent line is 10:

$$\frac{dy}{dx} = (x^2 - 3)' = 2x - 0$$

$$2(5) = 10$$

Hard way: Pick say, $(5.001, 5.001^2 - 3)$ and find slope of secant.

and see that it's close to 10.

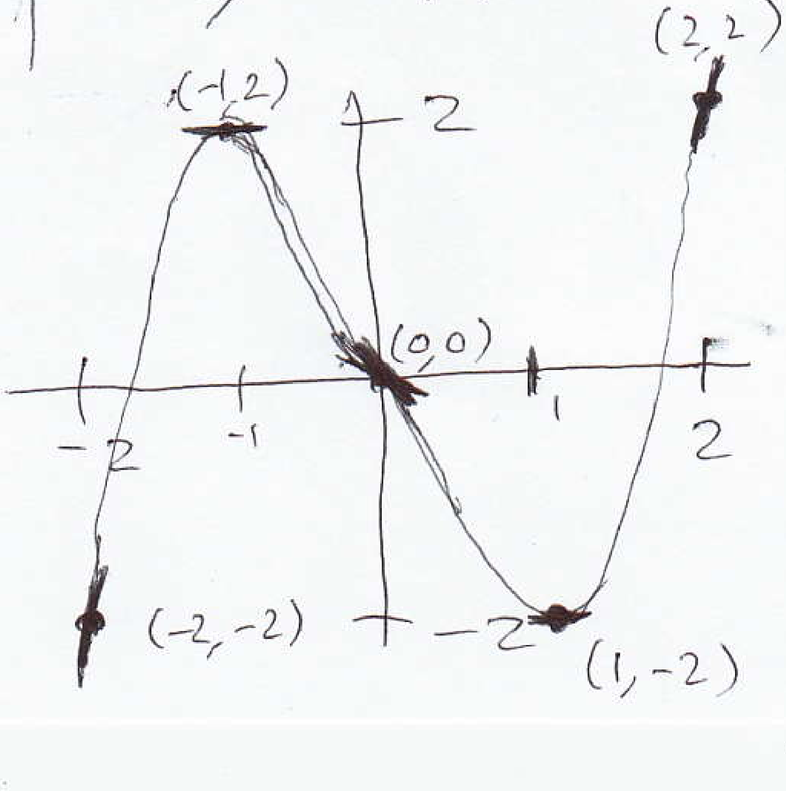


$y = x^3 - 3x$

$\frac{dy}{dx} = y' = (x^3 - 3x)'$

$\frac{dy}{dx} = 3x^2 - 3(1)$

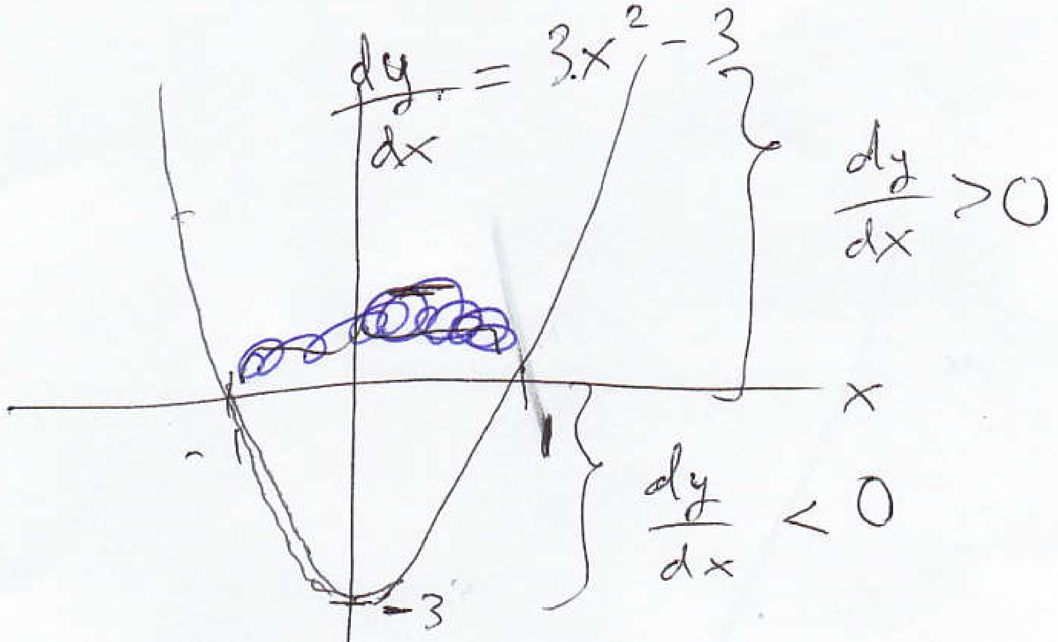
x	y	$dy/dx = 3x^2 - 3$
-2	-2	9
-1	2	0
0	0	-3
1	-2	0
2	2	9



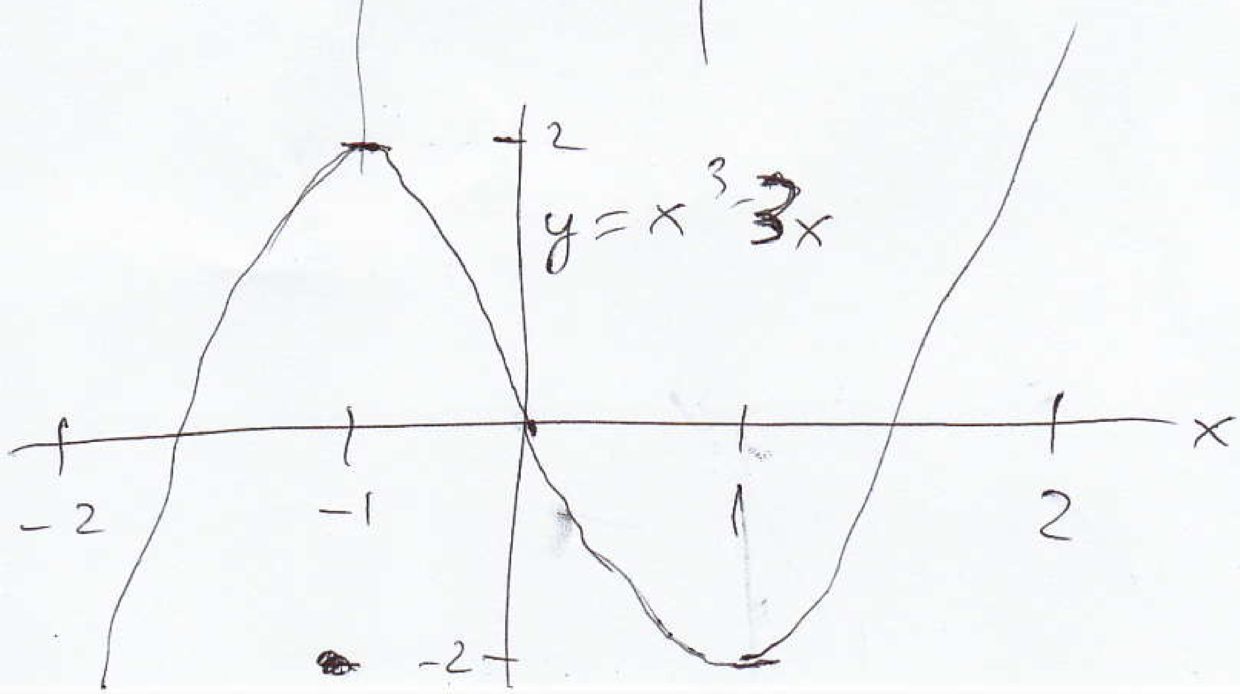
+ slope

− slope

0 slope



$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} > 0$
$y = x^3 - 3x \nearrow$	$x^3 - 3x \searrow$	$x^3 - 3x \nearrow$



derivatives are (instantaneous) rates of change. (Think of x-axis as time...)

Your speedometer is the derivative of your odometer.

$\boxed{\text{HW } 10\text{-}5 \ \# 50, 87, 88}$

(Optional reading)

Why is $d\left(\frac{f}{g}\right) = \frac{(df)g - f(dg)}{g^2}$?

Let $y = \frac{f}{g}$. Then $yg = f$, so

so $df = d(yg) \underset{\uparrow}{=} (dy)g + y(dg)$, so

product rule

$$df = \underbrace{\left(d\left(\frac{f}{g}\right)\right)}_{dy} g + \underbrace{\frac{f}{g}}_{y} dg,$$

Solve for $d\left(\frac{f}{g}\right)$:

$$df - \frac{f}{g}\,dg = \left(d\left(\frac{f}{g}\right)\right) g$$

$$\frac{1}{g}\left(df - \frac{f}{g}\,dg\right) = d\left(\frac{f}{g}\right)$$

$$\frac{df}{g} - \frac{f\,dg}{g^2} = d\left(\frac{f}{g}\right)$$

$$\frac{g\,df}{g^2} - \frac{f\,dg}{g^2} = d\left(\frac{f}{g}\right)$$

$$\boxed{\frac{(df)g - f(dg)}{g^2} = d\left(\frac{f}{g}\right)}$$