- a bit more on logarithms
- implicit differentiation (11-5)
- ~~related rates (11-6)~~

Exponential growth/decay of X:

$$dX = kX\,dt \qquad (t = \text{time})$$

k constant

Solution: $\boxed{X(t) = X(0)\,e^{kt}}$

$$\frac{dX}{X} = k\,dt \Rightarrow d(\ln X) = d(kt)$$

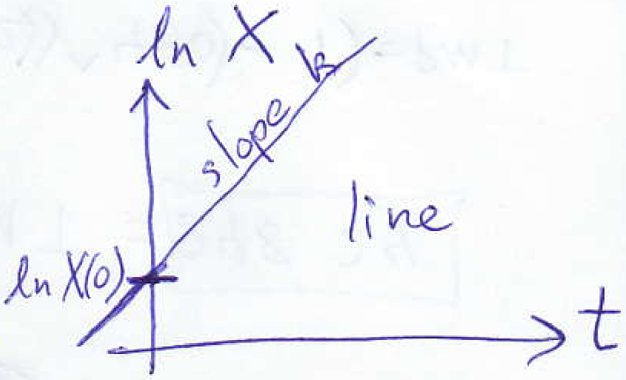
$$\Rightarrow d(\ln X) - d(kt) = 0 \Rightarrow d(\ln X - kt) = 0$$

$\Rightarrow \ln X - kt = \text{constant}$ Plug in $t=0$:

$$\ln(X(0)) - \underbrace{k\cdot(0)}_{0} = \text{constant}$$

$$\ln X - kt = \ln(X(0))$$

$$\boxed{\ln X = \ln(X(0)) + kt}$$



Exponential growth X is linear growth of $\ln X$.

$$\cancel{e}^{\ln X} = \cancel{e}^{\ln X(0) + kt} = \cancel{e}^{\ln X(0)} e^{kt}$$

$$\cancel{X =} \boxed{X(t) = X(0)\, e^{kt}}$$

If the US population is 312 million now (2011) and exponentially growing at 0.96% per year, estimate the population in 2025.

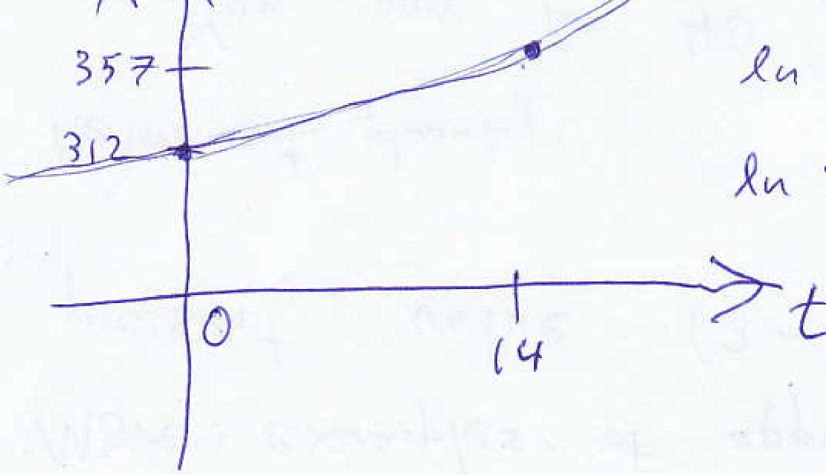
$t = 0$ in 2011 $\quad X(0) = 312$ (million)

$t = 14$ in 2025 $\quad X(14) = ?$

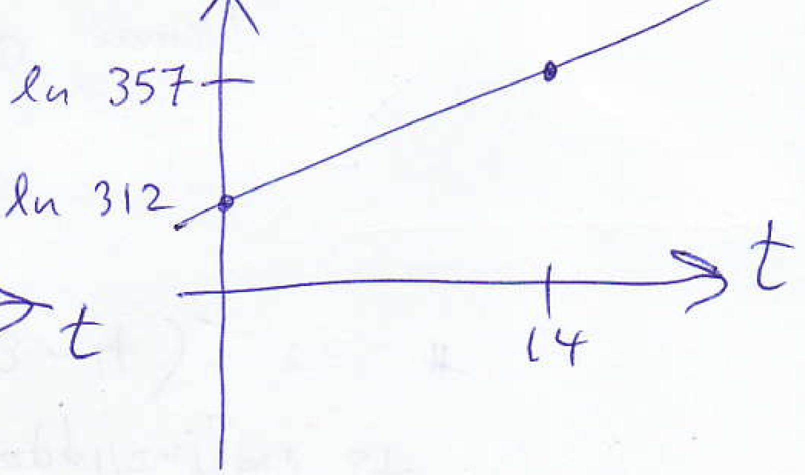
$k = 0.96\% = 0.0096$

$X(14) = X(0)\, e^{0.0096 \cdot 14} = 356.88\ldots$ million

$X = X(0)\, e^{0.0096t}$

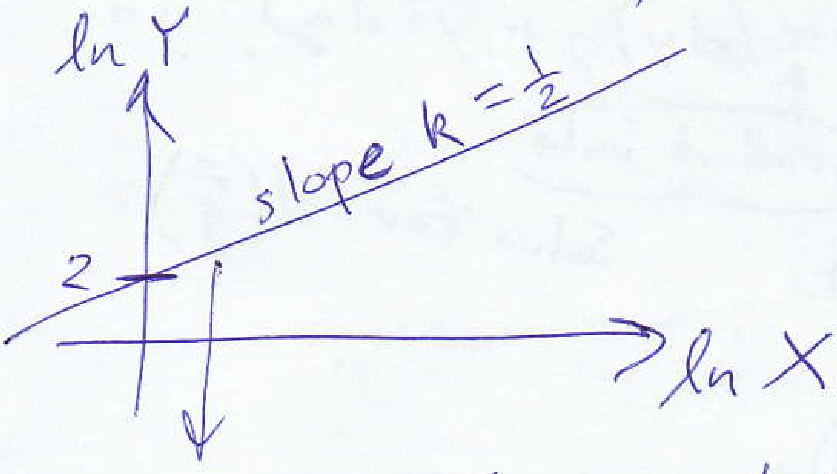


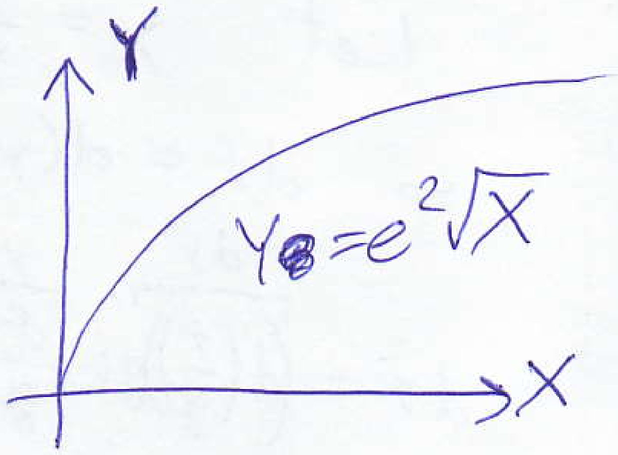
$\ln X = \ln(X(0)) + 0.0096t$



Constant elasticity: $\dfrac{dY/Y}{dX/X} = k = \text{constant}$

$$\frac{d(\ln Y)}{d(\ln X)} = \frac{dY/Y}{dX/X}$$





line: $\ln Y = 2 + \frac{1}{2} \ln X$

$$\cancel{e^{\ln}}\, Y = e^{2 + \frac{1}{2}\ln X} = e^{2} e^{\frac{1}{2}\ln X}$$

$$Y = e^{2}\, \cancel{e^{\ln}}(X^{1/2}) = e^{2}\sqrt{X}$$

<u>HW</u> If a family of four with household income of \$30,000 spends \$9,000 on food (per year) and later spends \$10,000 on food when income is higher at \$40,000 (per year),

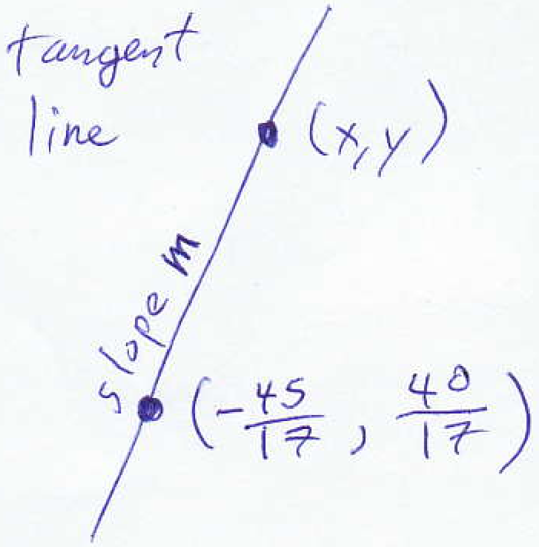
predict the family's annual food spending were their annual income $100,000, assuming constant elasticity.

Plot food spending vs. income with income ranging from 20,000 to 200,000.

Ellipse $\frac{x^2}{9}+\frac{y^2}{25}=1$

Point: $(-45/17, 40/17)$

Equation for tangent line: ?



$$\frac{\Delta y}{\Delta x}=m$$

$$\frac{y-40/17}{x-(-45/17)}=m$$

$$y-\frac{40}{17}=m\left(x+\frac{45}{17}\right)$$

But what's m?

It's dy/dx

$$\frac{x^2}{9} + \frac{y^2}{25} = 1 \Rightarrow d\left(\frac{x^2}{9} + \frac{y^2}{25}\right) = d(1)$$

$$d\left(\frac{x^2}{9}\right) + d\left(\frac{y^2}{25}\right) = 0$$

$$\frac{1}{9} d(x^2) + \frac{1}{25} d(y^2) = 0$$

$$\frac{1}{9} 2x\,dx + \frac{1}{25} 2y\,dy = 0$$

$$\frac{1}{9} 2x + \frac{1}{25} 2y \frac{dy}{dx} = 0$$

$$\frac{1}{25} 2y \frac{dy}{dx} = -\frac{1}{9} 2x$$

$$9 \cdot \cancel{2}y \frac{dy}{dx} = -25 \cdot \cancel{2}x$$

$$y \frac{dy}{dx} = -\frac{25}{9} x$$

$$\frac{dy}{dx} = \frac{-25x}{9y}$$

At $(x,y) = \left(-\frac{45}{17}, \frac{40}{17}\right)$: $\frac{dy}{dx} = \frac{25}{8} = 3.125$

tangent line: $y - \frac{40}{17} = \frac{25}{8}\left(x + \frac{45}{17}\right)$

$$0 = \underbrace{\left(x^4 - 3x^2 + x + y^4 - 4y^2 - 2y + 1\right)}_{f}\underbrace{\left(3(x-1)^2 + (2y-3)^2 - 1\right)}_{g}$$

$$\underbrace{d0}_{0} = d(fg) = df \cdot g + f \cdot dg$$

$$df = 4x^3\,dx - 3(2x\,dx) + 4y^3 dy - 4(2y\,dy)$$
$$-2dy + 0 + dx$$

$$dg = 3 \cdot \underbrace{2(x-1)\,d(x-1)}_{du^2 = 2u\,du} + \underbrace{2(2y-3)\,d(2y-3)}_{du^2 = 2u\,du} - 0$$

$$du^n = nu^{n-1}du$$

$$\frac{0}{dx} = \frac{df \cdot g + f \cdot dg}{dx}$$

$$0 = \frac{df}{dx}g + f\frac{dg}{dx}$$

$$\frac{dg}{dx} = \frac{6(x-1)\,dx}{dx} + \frac{(4y-6)(2dy)}{dx}$$

$$df/dx = 4x^3 - 6x + 4y^3\frac{dy}{dx} - 8y\frac{dy}{dx}$$
$$-2\frac{dy}{dx} + 1$$

Plug in $(x,y)=(0,-1)$.

(Only do this __after__ computing all differentials.)

$$df/dx = -4\frac{dy}{dx} + 8\frac{dy}{dx} - 2\frac{dy}{dx} + 1$$

$$df/dx = 2\,dy/dx + 1$$

$$dg/dx = 6(0-1) + (4(-1)-6)(2\,dy/dx)$$

$$dg/dx = -6 - 20\,dy/dx$$

$$f = 1 - 4 + 2 + 1 = 0$$

$$g = 3(-1)^2 + (-2-3)^2 - 1 = 27$$

$$0 = \frac{df}{dx}g + f\frac{dg}{dx}$$

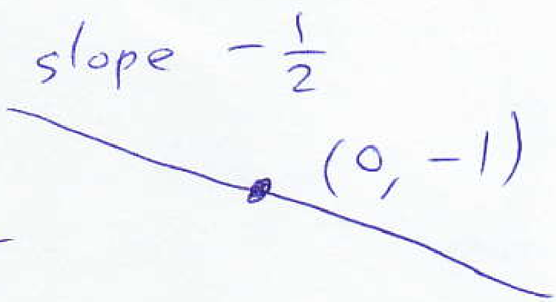
$$0 = \left(2\frac{dy}{dx} + 1\right)27 + \underbrace{0\left(-6-20\frac{dy}{dx}\right)}_{0}$$

$$0 = 2\frac{dy}{dx} + 1$$

$$-1 = 2\,dy/dx$$

$$-1/2 = dy/dx$$

slope of tangent line



tangent line: $y-(-1) = -\frac{1}{2}(x-0)$

If $0 \le m \le n$, then (except $m=n=0$)

$$(x,y) = \left(\pm \frac{a(m^2-n^2)}{m^2+n^2}, \pm \frac{2mnb}{m^2+n^2} \right)$$

is on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

HW: Pick 3 values for (m,n)

(e.g. $(m,n) = (2,7), (3,4), (1,2)$)

and sketch the ellipse

$\boxed{\begin{array}{l} a=7 \\ b=4 \end{array}}$ $\frac{x^2}{7^2} + \frac{y^2}{4^2} = 1$ with the

corresponding points on the ellipse and their tangent lines. Find equations for the tangent lines.

Aside: $(p^2-q^2)^2 + (2pq)^2 = (p^2+q^2)^2$

is why it works.