Quiz (3.1)

The units of momentum is:

A) $kg \cdot m^2/s^2$ B) $kg \cdot m/s$

C) $kg \cdot m^2/s^3$ D) $kg \cdot m/s^2$

Exam on 3/10 (Thur):

- Chapter 2 (2.1, 2.3, 2.4, 2.5)
- Also 3.1

(Notes + calculator OK)

Thur 3/3: • HW (Ch. 2, #31 — 34) is due

• Discuss 3.1 • No quiz

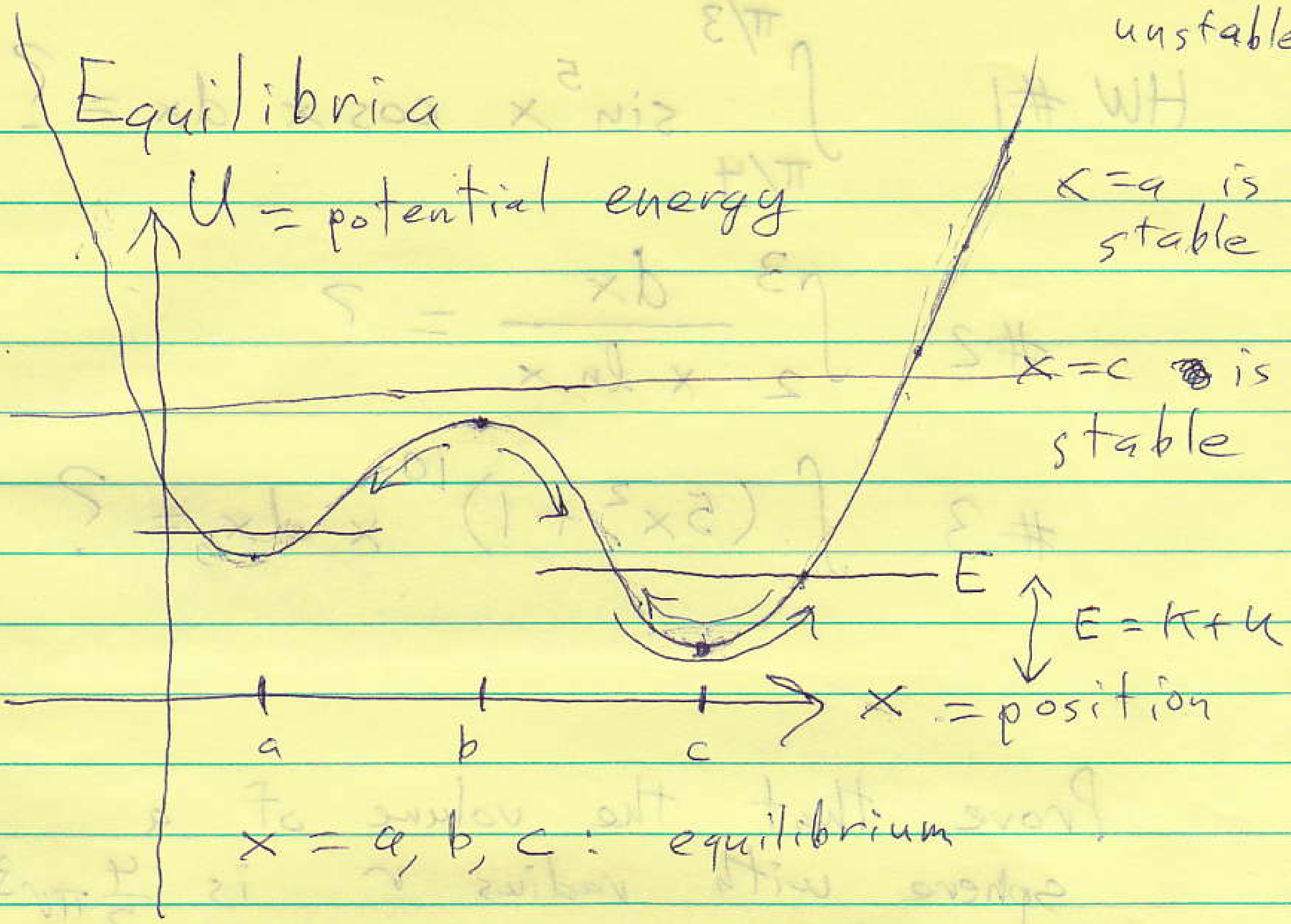
~~No quiz though.~~

Tues 3/8: • HW (Ch. 3, TBA) due

• no quiz

• Review

Equilibria



x = b is unstable

x = a is stable

x = c is stable

Suppose $U = Ax^4 - Bx^2$

$A = 1 \frac{J}{m^4} \qquad B = 2 \frac{J}{m^2}$

Where are the equilibria?

Where $\underbrace{\frac{dU}{dx}}_{} = 0$

slope of the U-x graph

$$U = Ax^4 - Bx^2$$

Where is $\frac{dU}{dx} = 0$?

$$d(x^n) = nx^{n-1}\,dx$$

$$\frac{dU}{dx} = \frac{A \cdot 4x^3\,dx - B \cdot 2x\,dx}{dx}$$

$$\frac{dU}{dx} = 4Ax^3 - 2Bx$$

Solve $0 = 4Ax^3 - 2Bx$

$$\cancel{0 = (4Ax^2 - 2B)}$$

$$0 = 2x(2Ax^2 - B)$$

$\alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta)$

$$0 = 2x(\sqrt{2Ax^2} + \sqrt{B})(\sqrt{2Ax^2} - \sqrt{B})$$

$\alpha - \beta = (\sqrt{\alpha} + \sqrt{\beta})(\sqrt{\alpha} - \sqrt{\beta})$

$0 = \alpha\beta\gamma \Rightarrow$ one of α, β, γ is 0

$2x = 0$ or $\sqrt{2Ax^2} + \sqrt{B} = 0$

or $\sqrt{2Ax^2} - \sqrt{B} = 0$

$2x = 0 \iff x = 0$ (1st equilibrium)

$\underbrace{\sqrt{2Ax^2}}_{\geq 0} + \underbrace{\sqrt{B}}_{>0}$ always positive

$0 = \sqrt{2Ax^2} - \sqrt{B} \iff \sqrt{2Ax^2} = \sqrt{B}$

$\iff \sqrt{2A}\,\sqrt{x^2} = \sqrt{B} \iff \sqrt{2A}\,(\pm x) = \sqrt{B}$

$\iff x = \pm \dfrac{\sqrt{B}}{\sqrt{2A}} = \pm \sqrt{\dfrac{B}{2A}}$

2nd & 3rd equilibria

$A = 1\ J/m^4$

$B = 2\ J/m^2$

$\pm\sqrt{\dfrac{B}{2A}} = \pm\sqrt{\dfrac{2\ J/m^2}{\cancel{1}\ J/m^4}} = \pm\sqrt{\dfrac{(\cancel{2}\ J/m^2)m^4}{(\cancel{1}\ J/m^4)m^4}}$

$= \pm\sqrt{\dfrac{2\ J\cdot m^2}{\cancel{1}\ J}} = \cancel{\pm\sqrt{2}\ m} = \cancel{\pm\sqrt{2}\ m}$

2

$= \pm\sqrt{1 m^2} = \pm 1 m.$

$U = Ax^4 - Bx^2$

3 equilibria: $x = -1, 0, +1$

Parabolic approximations of U near each equilibrium

For x near some value a,

$$U \approx P + Q(x-a) + \frac{1}{2}R(x-a)^2$$

where $P = U$ at $x=a$

$Q = \frac{dU}{dx}$ at $x=a$

$R = \frac{d^2U}{dx^2}$ at $x=a$

If $x=a$ is an equilibrium then $Q = \frac{dU}{dx} = 0$.

Let's find $R = \dfrac{d^2U}{dx^2}$ at $\underline{x = 1\,m}$.

$$dU = 4Ax^3\,dx - 2Bx\,dx$$

$$d(dU) = d^2U = d(4Ax^3)\,dx - d(2Bx)\,dx$$

$$= (4A \cdot 3x^2\,dx)\,dx - (2B\,dx)\,dx$$

$$= (12Ax^2 - 2B)\,dx^2$$

$$\frac{d^2U}{dx^2} = 12Ax^2 - 2B$$

At $x = 1\,m$, $\dfrac{d^2U}{dx^2} = 12A(1\,m)^2 - 2B$

$$= 12\left(\frac{J}{m^4}\right)m^2 - 2\left(\frac{2J}{m^2}\right)$$

$$= 8\ J/m^2$$

$$\underbrace{(4Ax^3 + 2Bx)'}_{J/m} = \underbrace{12Ax^2 - 2B}_{J/m^2}$$

We also need $P = U$ at $x = 1\,m$:

$$U = A(1m)^4 - B(1m^2)$$

$$= \left(1\frac{J}{\cancel{m^4}}\right)\cancel{m^4} - \frac{2J}{\cancel{m^2}}\cancel{m^2} = -1J$$

Parabolic approximation of U

near $x = 1\,m$:

$$U \approx P + Q(x - 1m) + \frac{1}{2}R(x - 1m)^2$$

$$P = -1J \qquad Q = 0$$

$$R = 16\,J/m^2$$

If you write this without the units:

$$U \approx -1 + 0 + \frac{1}{2}(\cancel{16})(x-1)^2$$

8

$$U \approx 4(x-1)^2 - 1$$

Near $x = 1\,m$ $\quad U = \left(\frac{1J}{m^4}\right)x^4 - \left(\frac{2J}{m^2}\right)x^2$

is ~~a~~ approximately parabolic:

$$U \approx \underbrace{-1J}_{c} + \underbrace{\left(\frac{4J}{m^2}\right)}_{\frac{1}{2}k}(x - 1m)^2$$

"Parabolic U w/ stable equilibrium"

has a name: simple harmonic oscillator.

$$\begin{cases} U = \frac{1}{2}k(x - x_{eq})^2 + c \\ \& \; K = \frac{1}{2}mv^2 \quad \& \quad E = K + U \end{cases}$$

oscillates with period

$$T = 2\pi\sqrt{\frac{m}{k}}$$
m ← mass
k ← effective spring constant

E.g. $T = 2\pi\sqrt{\frac{\text{mass}}{8J/m^2}}$ in our example

Another example:

$$U = 5 + 6x + 7x^2$$

Approximate U near $x = 0$

$$\frac{dU}{dx} = 6 + 14x = 6 \text{ at } x = 0$$

$$\frac{d^2U}{dx^2} = 14 = 14 \text{ at } x = 0$$

$$U = 5 \text{ at } x = 0$$

$$U = 5 + 6x + \frac{1}{2} \cdot 14 \, x^2$$

(Perfect approximation because U was already a parabola.)