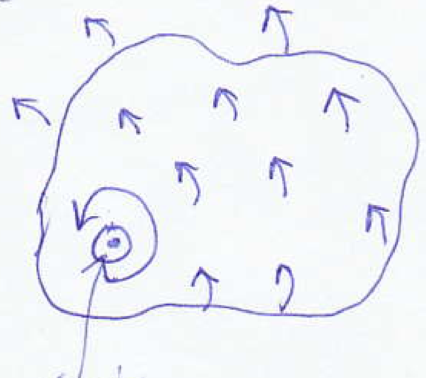
Rotation of a rigid body about a single axis (Section 4.2):



axis

T = period = time per rotation

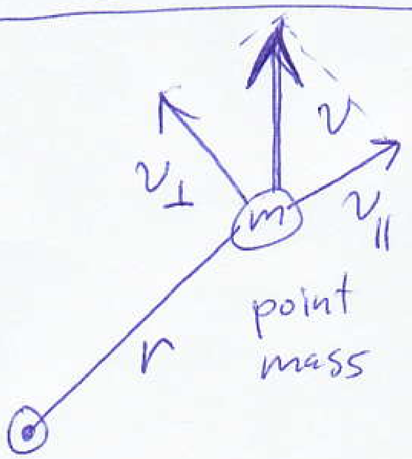
ω = "angular frequency"

ω = "angular velocity"

$\omega = \frac{d\theta}{dt}$; θ = angular position

<u>If</u> ω is constant, then $\omega = \frac{2\pi}{T}$.

Start with simple case:



$v_\perp = \omega r$ ← (similar to circular motion)

$v_\perp$ is the rotational part of mass m's velocity.

$v_\parallel$ is the "translational" part.

point mass: $L = mrv_\perp = mr\omega r = mr^2\omega = I\omega$

I = moment of inertia = mr^2 (definition of I)

<u>If</u> $v_\parallel = 0$, then I is constant, so

$$\tau = \frac{dL}{dt} = I\frac{d\omega}{dt} = I\alpha \quad \left(\alpha = \frac{d\omega}{dt} = \text{angular acceleration}\right)$$

If $v_{\parallel}=0$, then $K=\frac{1}{2}mv^2=\frac{1}{2}mv_{\perp}^2$, so

$$K=\frac{1}{2}m(\omega r)^2=\frac{1}{2}mr^2\omega^2=\frac{1}{2}I\omega^2.$$

If $v_{\parallel}\neq 0$, then $K=\frac{1}{2}mv^2=\frac{1}{2}m(v_{\perp}^2+v_{\parallel}^2)$,

so $$K=\underbrace{\frac{1}{2}mv_{\parallel}^2}_{K_{tran}}+\underbrace{\frac{1}{2}I\omega^2}_{K_{rot}}.$$

"Translational" kinetic energy

"Rotational" kinetic energy

For a rigid body, we add up all the I's, L's, K's, etc. for each particle in the body: $I=m_1r_1^2+m_2r_2^2+\cdots+m_Nr_N^2$

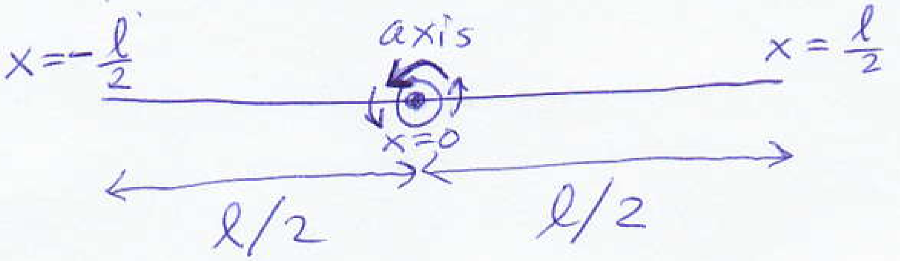
$$L=m_1r_1^2\omega+m_2r_2^2\omega+\cdots+m_Nr_N^2\omega=I\omega$$

$$K=\frac{1}{2}m_1r_1^2\omega^2+\cdots+\frac{1}{2}m_Nr_N^2\omega^2=\frac{1}{2}I\omega^2$$

(All particles in a rigid body must have the same ω, assuming there is only rotation about the chosen axis.)

When N is large, it often is better to treat the body as continuous:

Thin rod of mass M and length ℓ:



$r = |x|$

$r^2 = x^2$

$$I = m_1 r_1^2 + \cdots + m_N r_N^2 \approx \int_{\text{rod}} r^2 \, dm$$

$$I = \int_{x=-\ell/2}^{x=\ell/2} x^2 \, dm \left(\frac{dx}{dx}\right) = \int_{-\ell/2}^{\ell/2} x^2 \left(\frac{dm}{dx}\right) dx$$

Assuming uniform density of the rod, $\frac{dm}{dx}$ is constant, so $\frac{dm}{dx} = \frac{M}{\ell}$, so $I = \left(\int_{-\ell/2}^{\ell/2} x^2 \, dx\right)\left(\frac{M}{\ell}\right)$

$$I = \left(\int_{-\ell/2}^{\ell/2} d(x^3)/3\right)\left(\frac{M}{\ell}\right) = \left(\frac{(\ell/2)^3}{3} - \frac{(-\ell/2)^3}{3}\right)\left(\frac{M}{\ell}\right)$$

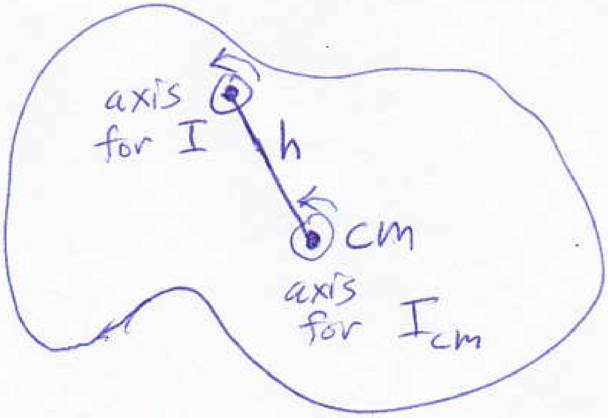
Note: $d(x^3) = 3x^2 dx$

$$I = \left(\frac{\ell^3/8}{3} - \frac{-\ell^3/8}{3}\right)\left(\frac{M}{\ell}\right) = \frac{2M\ell^2/8}{3} = \frac{M\ell^2}{12}$$

"Translational" motion	Rotation of rigid body about single axis
$v = \frac{dx}{dt}$, $a = \frac{dv}{dt}$	$\omega = \frac{d\theta}{dt}$, $\alpha = \frac{d\omega}{dt}$
$F = \frac{dp}{dt} = ma$	$\tau = \frac{dL}{dt} = I\alpha$
$p = mv$	$L = I\omega$
$K_{tran} = \frac{1}{2}mv^2$	$K_{rot} = \frac{1}{2}I\omega^2$
When a is constant, $\Delta x = \frac{1}{2}a(\Delta t)^2 + v\Delta t$.	When α is constant, $\Delta\theta = \frac{1}{2}\alpha(\Delta t)^2 + \omega\Delta t$.
When v is constant, $\Delta x = v\Delta t$.	When ω is constant, $\Delta\theta = \omega\Delta t$.
a constant $\Rightarrow \Delta(v^2) = 2a\Delta x$	α constant $\Rightarrow \Delta(\omega^2) = 2\alpha\Delta\theta$

Your book has formulas for moments of inertia of common shapes (about common choices of axes). (Page 268.)

You can combine these formulas with the Parallel Axis Theorem to compute even more moments of inertia without having to do a new integral: $I = I_{cm} + Mh^2$



cm = center of mass

I_{cm} = moment of inertia for axis through cm

$$I = I_{cm} + Mh^2$$

h = distance between axes

total mass = M

$$cm = (x_{cm}, y_{cm})$$

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_N x_N}{m_1 + m_2 + \cdots + m_N}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + \cdots m_N y_N}{m_1 + m_2 + \cdots + m_N}$$

$$M = m_1 + m_2 + \cdots + m_N$$

Continuous version:

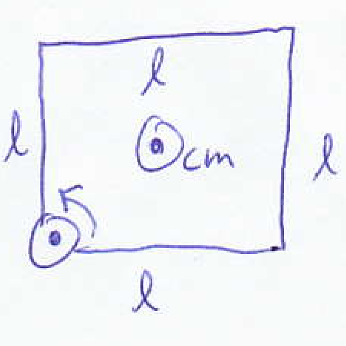
$$x_{cm} = \frac{\int x\, dm}{\int dm} = \frac{\int x\, dm}{M}$$

$$y_{cm} = \frac{\int y\, dm}{\int dm} = \frac{\int y\, dm}{M}$$

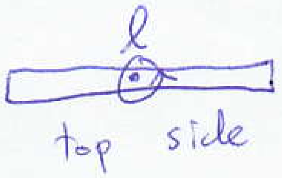
$$M = \int dm$$

Bend thin rod of mass M & length 4ℓ into square & find I for axes through

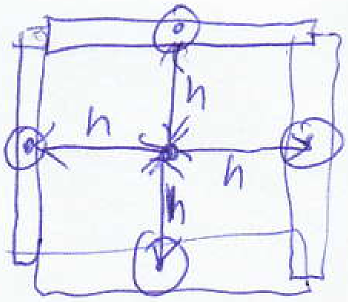
a corner:



~~First, find I_{cm} by adding~~

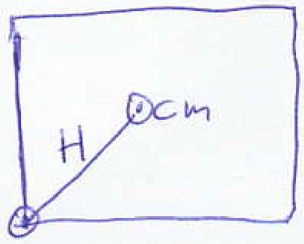


mass $M/4$: $I^{top}_{cm} = \frac{(M/4)\ell^2}{12} = \frac{M\ell^2}{48}$

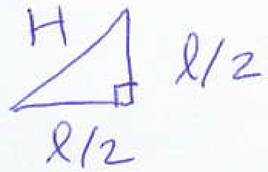


$\left(h = \frac{\ell}{2}\right)$ For I_{cm} for the whole square, each side contributes $\frac{M\ell^2}{48} + \left(\frac{M}{4}\right)h^2$, so

$$I_{cm} = 4\left(\frac{M\ell^2}{48} + \left(\frac{M}{4}\right)\left(\frac{\ell}{2}\right)^2\right) = \frac{M\ell^2}{12} + \frac{M\ell^2}{4} = \frac{M\ell^2}{3}$$



$$I = I_{cm} + MH^2 = \frac{M\ell^2}{3} + \frac{M\ell^2}{2}$$



$$\left(\frac{\ell}{2}\right)^2 + \left(\frac{\ell}{2}\right)^2 = H^2$$

$$\frac{\ell^2}{2} = \frac{\ell^2}{4} + \frac{\ell^2}{4} = H^2$$

$$I = \frac{5}{6}M\ell^2$$

HW due April 12 (Tuesday):

1. Chapter 4, #22

Read 4.3 before next class on April 12.

HW (2) Find the center of mass

of the following 3 point masses:

$m_1 = 1.5\,kg$ $\quad$ $m_2 = 0.62\,kg$ $\quad$ $m_3 = 1.5\,kg$

$x_1 = 3.2\,m$ $\quad$ $x_2 = 0.0\,m$ $\quad$ $x_3 = 0.75\,m$

$y_1 = 1.1\,m$ $\quad$ $y_2 = -2.0\,m$ $\quad$ $y_3 = 0.0\,m$

Then find I_{cm} and I_1, the moments of inertia for the axes going through the c.m., and mass m_1, respectively.

HW (3) ~~Find the~~ If a solid cylinder of radius 30cm is rolling at a rate of 25 revolutions per minute, and its mass is 75kg, then what is its kinetic energy (in joules)?

~~Hint: First work in reference frame of center of the cylinder to find the rotational kinetic energy. Then go back to the "ground" frame to find the translational kinetic energy.~~

HW 4 A solid sphere of mass 400g and radius 15mm is spinning on a table, initially at a rate of 6.5 revolutions per second. After 45 seconds, the sphere comes to rest. What was the average torque (due to friction)?