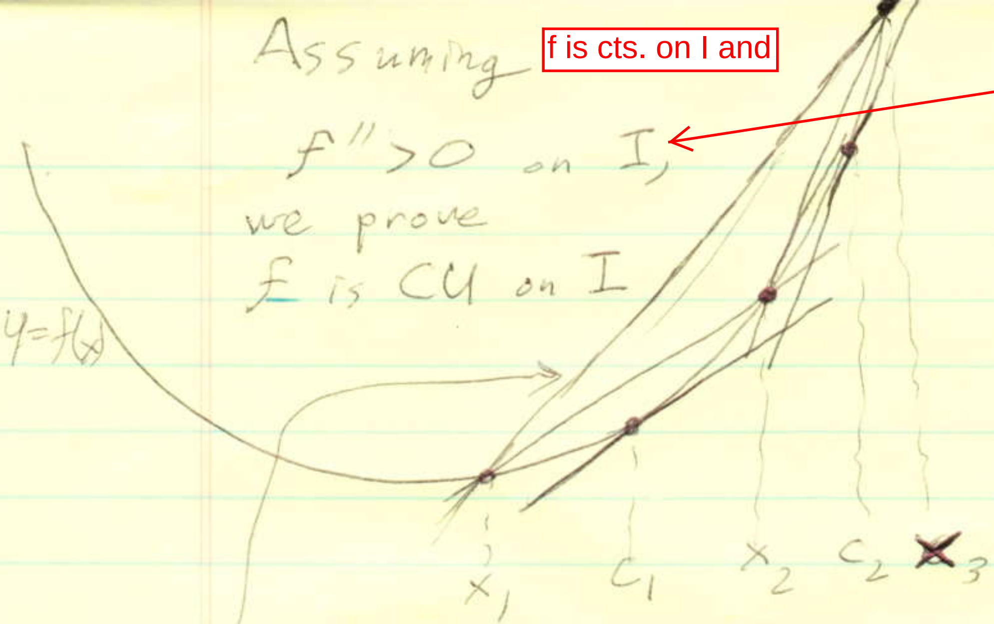
Assuming f is cts. on I and

$f'' > 0$ on $I$, or just the interior of I

we prove

$f$ is CU on $I$




MVT:

$$f'(c_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f'(c_2) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

$$y = \ell(x) = \frac{f(x_3) - f(x_1)}{x_3 - x_1}(x - x_1) + f(x_1)$$

Let us prove $\ell(x_2) > f(x_2)$:
We just need to show that

$$\ell(x_2) - f(x_2) > 0.$$

We will use the fact that since
$f'' > 0$ on $I$, $f'$ is $\nearrow$ on $I$,
so $f'(c_1) < f'(c_2)$.

Also, we will need to rearrange the above two instances of the MVT:

$$f'(c_1)(x_2 - x_1) = f(x_2) - f(x_1)$$
$$f'(c_2)(x_3 - x_2) = f(x_3) - f(x_2)$$

$$\ell(x_2) - f(x_2) = \frac{f(x_3)-f(x_1)}{x_3-x_1}(x_2-x_1) + f(x_1) - f(x_2)$$

$$= \frac{(f(x_3)-f(x_1))(x_2-x_1) + (f(x_1)-f(x_2))(x_3-x_1)}{x_3-x_1}$$

$$= \frac{[(f(x_3)-f(x_2)) + (f(x_2)-f(x_1))](x_2-x_1) + \underbrace{(f(x_1)-f(x_2))}_{-(f(x_2)-f(x_1))}(x_3-x_1)}{x_3-x_1}$$

$$\boxed{(x_2-x_1) - (x_3-x_1) = -(x_3-x_2)}$$

$$= (x_3-x_1)^{-1}\left[(f'(c_2)(x_3-x_2) + f'(c_1)(x_2-x_1))(x_2-x_1) - f'(c_1)(x_2-x_1)(x_3-x_1)\right]$$

$$= (x_3-x_1)^{-1}\left[f'(c_2)(x_3-x_2)(x_2-x_1) - f'(c_1)(x_2-x_1)(x_3-x_2)\right]$$

$$= \frac{(f'(c_2)-f'(c_1))(x_2-x_1)(x_3-x_2)}{x_3-x_1} > 0$$

because $f'(c_2) > f'(c_1)$, $x_2 > x_1$, $x_3 > x_2$, $x_3 > x_1$