Chain rule (2.6)

A bar of metal of length L is expanding as the temperature increases. Assume that right now the bar is expanding at $3 \cdot 10^{-6}$ m/°C and the temperature is increasing at $8 \cdot 10^{-2}$ °C/s. How fast is the bar expanding right now?

L - Length
T - Temperature
t = time

At this instant:

$$\frac{dL}{dT} = 3 \cdot 10^{-6} \text{ m/°C}$$

$$\frac{dT}{dt} = 8 \cdot 10^{-2} \text{ °C/s}$$

Looking for:
$\frac{dL}{dt} = ?$

$$\underbrace{\frac{dL}{dt} = \frac{dL}{dT} \cdot \frac{dT}{dt}}_{\text{chain rule}} = 3 \cdot 10^{-6} \text{ m/°C} \cdot 8 \cdot 10^{-2} \text{ °C/s}$$
$$= 24 \cdot 10^{-8} \frac{\text{m}}{\text{s}}$$

For average rate of change,

$$\frac{\Delta L}{\Delta T} = \frac{\Delta L}{\Delta T} \cdot \frac{\Delta T}{\Delta t} \quad \text{if } \Delta T \neq 0$$

Assume $y = f(u)$ & $u = g(x)$, and $f'(u)$ & $g'(x)$ exist.

Then $y = f(g(x))$ and $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, or $\left[\text{Chain Rule}\right]$

Equivalently: $(f(g(x)))' = f'(u)\, g'(x) = f'(g(x))\, g'(x)$

$(e^{\tan x})' = ?$ $\quad y = e^u \quad u = \tan x$

$y = e^{\tan x}$ and

$$(e^{\tan x})' = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (e^u)'(\tan x)'$$
$$= e^u \sec^2 x$$
$$= e^{\tan x}\sec^2 x$$

$(a^u)' = a^u \ln a$

a constant

$(e^u)' = e^u \underbrace{\ln e}_{1} = e^u$

$(\cos(x^2))' = ?$ $\quad y = \cos u$ & $u = x^2 \Rightarrow y = \cos(x^2)$

$$(\cos(x^2))' = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (\cos u)'(x^2)' = -\sin u \cdot 2x$$
$$= -2x\sin u = -2x\sin(x^2)$$

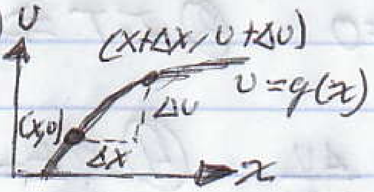
Why can't we cancel the du's in $\frac{dy}{du}\cdot\frac{du}{dx}$?

$\frac{dy}{du}\cdot\frac{du}{dx}$

$y = f(u)$	$u = g(x)$
$du = \Delta y$	$dx = \Delta x$
is any nonzero infinitesimal	is any nonzero infinitesimal
	$du = g'(x)dx$
	If $g'(x) = 0$ then $du = 0$

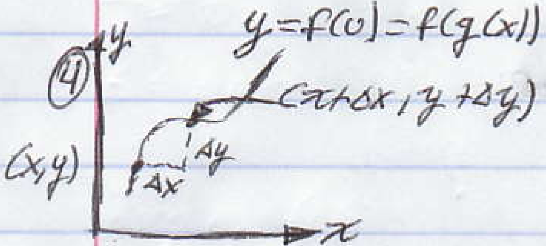
Here's how to get the chain rule without dividing by 0.

① $0 \neq \Delta x \approx 0$

② 

③ $u+\Delta u = g(x+\Delta x)$

$u = g(x)$

④ 

⑤ $y+\Delta y = f(g(x+\Delta x))$

$= f(g(x))$

Subtract: $\Delta y = \underbrace{f(g(x+\Delta x)}_{u+\Delta u} - \underbrace{f(g(x))}_{u}$

$\Delta y = f(u+\Delta u) - f(u)$

⑥ Since $g'(x)$ exists & $\Delta x \approx 0$, the Increment Theorem says $\Delta u \approx 0$ (and says more)

⑦ Since $\Delta u \approx 0$ and $f'(u)$ exists, then Increment Theorem says $\Delta y \approx 0$ and ε

$\Delta y = (f'(u) + \varepsilon)\Delta u$

for some $\varepsilon \approx 0$

⑧ $\frac{\Delta y}{\Delta x} = (f'(u)+\varepsilon)\frac{\Delta u}{\Delta x}$

⑨ Round to nearest real:

$\frac{dy}{dx} = \underbrace{(f'(u)+0)}_{\frac{dy}{du}}\frac{du}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

In the special case where $\Delta u \neq 0$, the proof is easier:

① $0 \neq \Delta x \approx 0$ ② I.T $\Rightarrow \Delta u \approx 0$

③ $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \frac{\Delta u}{\Delta x}$ ④ $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

More chain rule Examples:
Using chain rule twice:

$y = f(u)$, $u = g(w)$, $w = h(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{dy}{du} \frac{du}{dw} \frac{dw}{dx}$$

$$\left(\sqrt{\ln(7+x^5)}\right)' = ?$$

$w = 7 + x^5$ $\quad \frac{dw}{dx} = 5x^4$

$u = \ln w$ $\quad \frac{du}{dw} = \frac{1}{w}$

$y = \sqrt{u}$ $\quad \frac{dy}{du} = \frac{1}{2\sqrt{u}}$

$$\left(\sqrt{\ln(7+x^5)}\right) = \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dw} \frac{dw}{dx}$$

$$\frac{1}{2\sqrt{\ln w}} \cdot \frac{1}{7+x^5} \cdot 5x^4$$

$$= \frac{1}{2\sqrt{\ln(7+x^5)}} \cdot \frac{1}{7+x^5} \cdot 5x^4 = \frac{5x^4}{2(7+x^5)\sqrt{\ln(7+x^5)}}$$

$(\sin(t\cdot 2^t))' = ? = \frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$

$x = t\cdot 2^t \Rightarrow \frac{dx}{dt} = \underbrace{t'\cdot 2^t + t\cdot(2^t)'}_{\text{product rule}} = 1\cdot 2^t + t\cdot 2^t \ln 2.$

$y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$

$(\cos x)(2^t + t\cdot 2^t\cdot \ln 2) = [\cos(t\cdot 2^t)](2^t + t\cdot 2^t\cdot \ln 2)$

$\left(\frac{x}{\sqrt{x^2+1}}\right)' = \frac{x'\sqrt{x^2+1} - x(\sqrt{x^2+1})'}{(\sqrt{x^2+1})^2}$

$u = x^2+1 \quad \frac{du}{dx} = 2x$

$y = \sqrt{u} \quad \frac{dy}{du} = \frac{1}{2\sqrt{u}}$

$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = \frac{1}{2\sqrt{u}}\cdot 2x = \frac{1}{2\sqrt{x^2+1}}\cdot 2x = \frac{x}{\sqrt{x^2+1}}$

$\frac{x}{\sqrt{x^2+1}} = \frac{x'\sqrt{x^2+1} - x(\sqrt{x^2+1})'}{(\sqrt{x^2+1})^2} = \boxed{\frac{1\sqrt{x^2+1} - x(x/\sqrt{x^2+1})}{x^2+1}}$