+1.

Continuity (3.4)

"$f(x)$ is continuous at c" means

$$x \approx c \Rightarrow f(x) \approx f(c)$$

Equivalent definitions

1. $\lim_{x \to c} f(x) = f(c)$

2. $0 \neq \varepsilon \approx 0 \Rightarrow st(f(c+\varepsilon)) = f(c)$
3. $0 \neq \Delta x \approx 0 \Rightarrow st(f(c+\Delta x)) = f(c)$
4. $0 \neq \delta \approx 0 \Rightarrow f(c+\delta) \approx f(t)$

Eg.

$f(t) = t^2$ is continuous at 5.

2. $0 \neq \varepsilon \approx 0 \Rightarrow st((5+\varepsilon))^2 = (5+0)^2 = 25 = 5^2 = f$
1. $\lim_{t \to 5} t^2 = 25 = 5^2$

There are 3 ways $f(x)$ can fail to be continuous at c:

1. $f(c)$ is undefined/DNE
2. $\lim f(x)$ DNE (does not exist) On Next PG.
3. $\lim_{x \to c} f(x) \neq f(c)$ on Next Pg

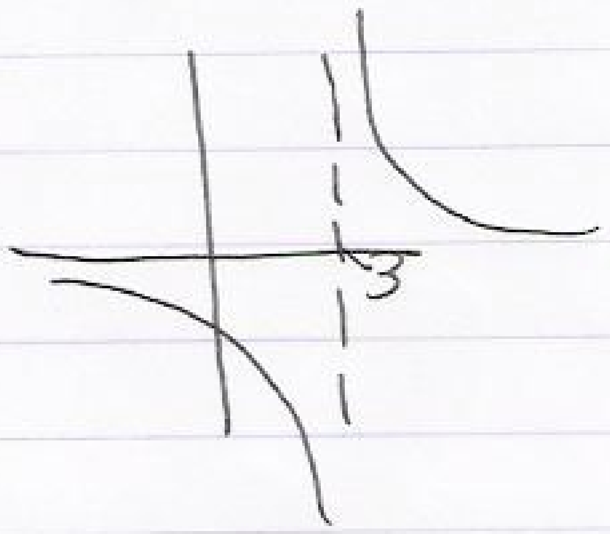
e.g.

$f(x) = \frac{x^2-1}{x-1}$ $\lim f(x) = st\left(\frac{(1+\varepsilon)^2-1}{1+\varepsilon-1}\right) = st\left(\frac{2\varepsilon+\varepsilon^2}{\varepsilon}\right) =$

$st(2+\varepsilon) = 2+0 = 2$ But $f(1) = \frac{1^2-1}{1-1} = \frac{0}{0}$ is undefined

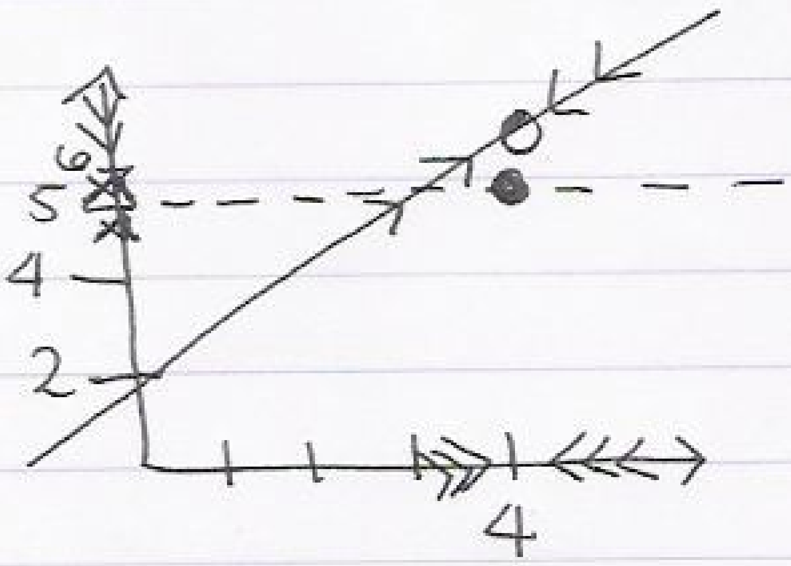
E.g. ② $f(x) = \frac{1}{x-3}$ $\lim\limits_{x \to 3} \frac{1}{x-3} = st\left(\frac{1}{3+\varepsilon-3}\right) = st\left(\frac{1}{\varepsilon}\right) \leftarrow$ DNE

$0 \neq \varepsilon \approx 0$ because $\frac{1}{\varepsilon}$ is infinite



E.g. ③

$$f(x)\begin{cases} x+2 & x \neq 4 \\ 5 & x = 4 \end{cases}$$



$\lim f(x) = st(f(4+\varepsilon)) = st(4+\varepsilon+2) = 4+0+2 = 6 \neq 5 = f(4)$

$0 \neq \varepsilon \approx 0$

continuous function are nice for limits:

$$\lim_{x \to c} f(x) = f(c)$$

So, which functions are continuous and which x-values?

<u>Fact</u>: If $f'(c)$ exists, then f is continuous at c.
Why? Let $0 \neq \Delta x \approx 0$

By definition, $st\left(\frac{f(c+\Delta x) - f(c)}{\Delta x}\right) = f'(c)$

$$\underbrace{st(\Delta x)}_{0}\, st\left(\frac{f(c+\Delta x)-f(c)}{\Delta x}\right) = f'(c)\,\underbrace{st(\Delta x)}_{0}$$

$$st\left(\frac{\cancel{\Delta x}\cdot f(c+\Delta x)-f(c)}{\cancel{\Delta x}}\right) = 0$$

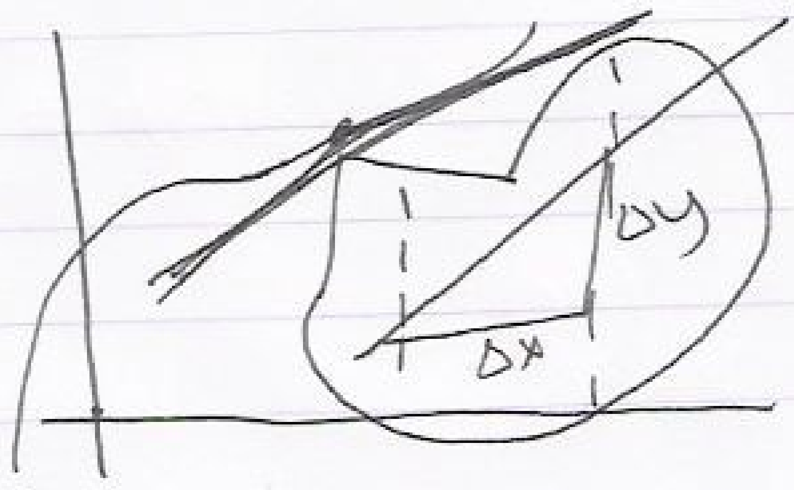
$$st\big(f(c+\Delta x)-f(c)\big) = 0$$

$$st\big(f(c+\Delta x)\big) - st\big(\underbrace{f(c)}_{\text{real}}\big) = 0$$

$$st\big(f(c+\Delta x)\big) - f(c) = 0$$

$$\boxed{st\big(f(c+\Delta x)\big) = f(c)} \quad \lim_{x\to c} f(x) = f(c)$$

Picture

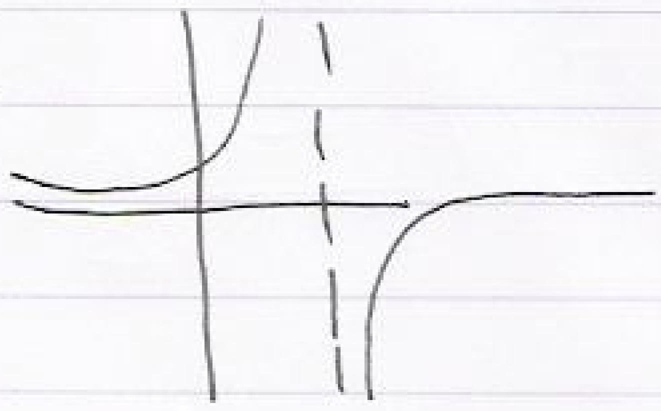


$$f(c+\Delta x) - f(c) = \Delta y \approx f'(c)\,\Delta x$$

$$\lim_{x\to \frac{\pi}{3}} \sin(x) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Sine is differentiable at $\pi/3$ (and everywhere else), so sine is continuous at $\pi/3$ (and everywhere else)

$\lim_{x \to 3} \frac{1}{x-3}$ DNE



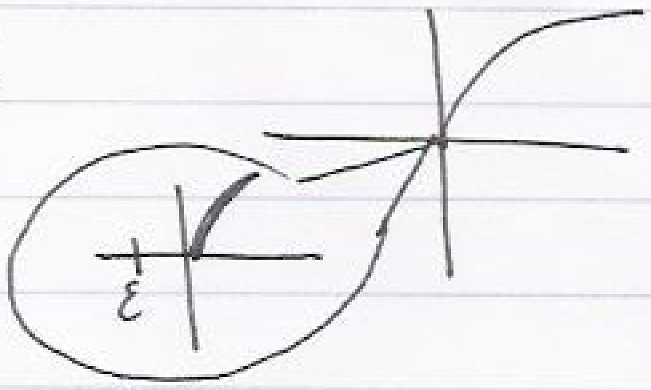
$\left(\frac{1}{x-3}\right)' = \frac{-1}{(x-3)^2}$ ← not defined at $x = 3$

$\lim_{x \to 5} \frac{1}{x-3} = \frac{1}{5-3} = \frac{1}{2}$

$\left(\frac{1}{x-3}\right)'$ exist at $5 = x$, so $\frac{1}{x-3}$ is cts. there

$\sqrt{x}$ is not cts. at 0 because

$\lim_{x \to 0} \sqrt{x}$ DNE



$\sqrt{0+\varepsilon}$ is not defined for $0 > \varepsilon \approx 0$, so $st(\sqrt{0+\varepsilon})$ is not defined,

$\lim_{x \to 0} \sqrt{x}$ DNE