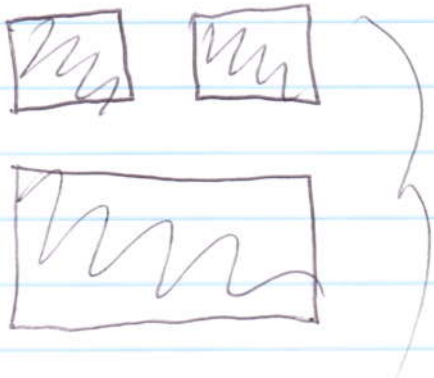
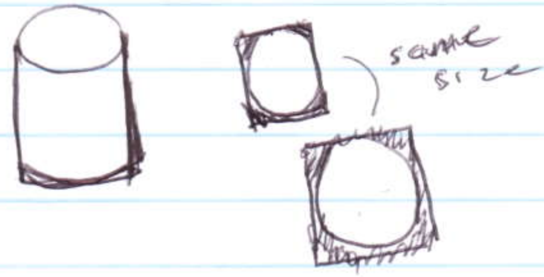
Optimization (3.6)

1. Draw a picture
2. Write a formula $Q = y(x, y, z)$ for quantity Q to be optimized.
3. Write formula(s) for constraints on independent variables $x, y, z, \ldots$
4. Use constraints to eliminate all but one independent variable: $Q = f(x)$.
5. Use a "single critical point method" to find max/min of $f(x)$.
6. Sometimes, it works fine to do step 5 over the interval of all numbers $(-\infty, \infty)$. Sometimes you need to restrict to a smaller interval. E.g. distances cannot be negative, so restrict to $[0, \infty)$.

A circular cylindrical aluminum can is to have volume 16 fl. oz. $= 473\,cm^3$. It is made cutting circles out of squares for the top and bottom sides. The curved sides is just a rolled rectangle. The extra aluminum from cutting out circles is recycled to recover $\frac{1}{5}$ of its cost. Find the dimensions of the can that minimizes cost (of aluminum).

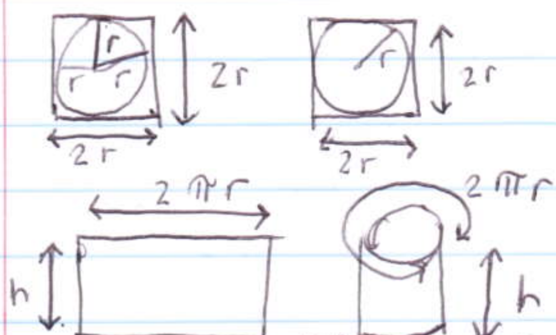




can + extras
recycled

radius = r
height = h } variable

a = cost of Al per unit area } constant



$$\text{area used} = \overbrace{(2r)^2}^{4r^2} + \overbrace{(2r)^2}^{4r^2} + 2\pi r h = 8r^2 + 2\pi r h$$

$$\text{area recycled} = \left[(2r)^2 - \pi r^2\right] + \left[(2r)^2 - \pi r^2\right] = 2\left[4r^2 - \pi r^2\right] = 8r^2 - 2\pi r^2$$

$$\text{cost} = a \cdot (\text{area used}) - \frac{1}{5} \cdot a \cdot (\text{area recycled}) =$$

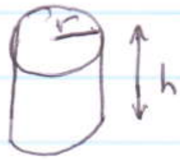
$$a \cdot (8r^2 + 2\pi r h) - \frac{a}{5}(8r^2 - 2\pi r^2)$$

$$\cancel{\frac{a}{5}} \quad \text{cost} = \frac{a}{5} \cdot 5(8r^2 + 2\pi r h) - \frac{a}{5}(8r^2 - 2\pi r^2) =$$

$$\frac{a}{5}\left((40 - 8 + 2\pi)r^2 + 10\pi r h\right) = \frac{2a}{5}\left((16 + \pi)r^2 + 5\pi r h\right)$$

Measure r and h in cm

$473\ cm^3 \underset{\text{of can}}{=\text{volume}} = \pi r^2 h$



constraint

$473 = \pi r^2 h$

$$\boxed{\frac{473}{\pi r^2} = h}$$

minimize this

$$\frac{2a}{5}\left((16+\pi)r^2 + 5\pi r h\right) = \frac{2a}{5}\left((16+\pi)r^2 + 5\pi r \cdot \frac{47}{\pi r^2}\right.$$

$$\text{cost} = C = \frac{2a}{5}\left((16+\pi)r^2 + \frac{2365}{r}\right)$$

$$C' = \frac{dC}{dr} = \frac{2a}{5}\left((16+\pi)2r + 2365 \cdot \left(\frac{-1}{r^2}\right)\right)$$

C' not defined at $r=0$.

We should restrict r to be positive:

r in $(0,\infty)$.

This excludes $r=0$ as critical point.

Solve $C'=0$

$$0 = \frac{2a}{5}\left((16+\pi)2r - \frac{2365}{r^2}\right)$$

$$0 = \frac{0}{2a/5} = (16+\pi)2r - \frac{2365}{r^2}$$

$$\frac{2365}{\cancel{r^2}} = (16+\pi)2r^3$$

$$\frac{2365}{(16+\pi)2} = r^3 \Rightarrow r = \sqrt[3]{\frac{2365}{32+2\pi}} = 3.95\ldots \text{ (cm)}$$

unique crit. pt in $(0,\infty)$

2nd Derivative Test will be easier.

$$C' = \frac{2a}{5}\left((16+\pi)2r - 2365r^{-2}\right)$$

$$C'' = \frac{2a}{5}\left((16+\pi)2 - 2365(-2)r^{-3}\right)$$

$$C'' = \underbrace{\frac{2a}{5}\left((16+\pi)^2 + 4730/r^3\right)}_{\text{positive if } r>0}$$

So, $C'' > 0$ at $r = 3.95$

so the minimum is at $r = 3.95\ldots$

Recall $\begin{cases} f''(\text{crit pt}) > 0 \Rightarrow \text{min} \\ f''(\text{crit pt}) < 0 \Rightarrow \text{max} \\ \text{otherwise test fails} \end{cases}$

We already found $h = \dfrac{473}{\pi r^2}$

The optimal $h = \dfrac{473}{\pi r^2}$

The optimal h is $\dfrac{473}{\pi\left(\underbrace{\sqrt[3]{\dfrac{2365}{32+2\pi}}}_{\text{do not round of to 3.95}}\right)^2} = \cancel{9.6497\ldots}$ 9.6345... cm

Webwork would accept ~~9.649~~ 9.634 but not ~~9.634~~

$$9.649\ldots = \frac{473}{\pi(3.95)^2}$$