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About 5 questions

1 sheet of Notes and Calculator.

Review Midterm II

10-26-10

Sections
2.4
2.5
2.6
2.7
2.8
3.2
3.3 +
3.4
3.8
3.5+
HW 5-8

3.2 Related Rates

A spherical balloon is losing volume, currently at a rate of 2 cm^3/sec if the volume is currently 100,000 cm^3, what is the rate of change of the radius of the balloon right now?

$V = \frac{4}{3}\pi r^3$ (true at all times)

$dV = d\left(\frac{4}{3}\pi r^3\right)$

$dV = \frac{4}{3}\pi\, d(r^3) = \frac{4}{3}\pi\, 3r^2\, dr$

$\frac{dV}{dt} = \frac{4\pi r^2 dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

Right now: $\frac{dV}{dt} = -2$ (losing volume)

$V = 100000 = 10^5$

$\frac{4}{3}\pi r^3 = 10^5$

$r^3 = \frac{3 \cdot 10^5}{4\pi} \Rightarrow r = \sqrt[3]{\frac{3\cdot 10^5}{4\pi}}$

$r^2 = \left(\frac{3\cdot 10^5}{4\pi}\right)^{2/3}$

$-2 = 4\pi \left(\frac{3\cdot 10^5}{4\pi}\right)^{2/3} \frac{dr}{dt}$

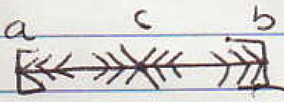
$$\boxed{\frac{-2}{4\pi\left(\frac{3\cdot 10^5}{4\pi}\right)^{2/3}} = \frac{dr}{dt}}$$

3.4 Continuity

f is continuous at c if $\lim_{x \to c} f(x) = f(c)$

continuous from left: $\lim_{x \to c^-} f(x) = f(c)$

continuous from right: $\lim_{x \to c^+} f(x) = f(c)$



continuous on $[a,b]$: $\left\{ \begin{array}{l} \lim_{x \to c} f(x) = f(c) \text{ for all } c \text{ in } (a,b) \\ \lim_{x \to a^+} f(x) = f(a) \\ \lim_{x \to b^-} f(x) = f(b) \end{array} \right.$

(for formulas built from "elementary functions")

You avoid, for all x in $[a,b]$, division by 0, even$\sqrt{\text{negative}}$, ln(0), ln(negative). $\Longleftrightarrow$

Is $\dfrac{\ln(x^2+1)}{x(1-3x)}$ continuous on $[0.1, 5]$?

$x^2 \geq 0 \Rightarrow x^2+1 \geq 1 > 0 \Rightarrow$ avoid ln(0) & ln(negative)

$x(1-3x) = 0 \Leftrightarrow x = 0$ or $1-3x=0$

$\Leftrightarrow x=0$ or $x = 1/3 = 0.333\ldots$

in $[0.1, 5]$

$\boxed{\text{NOT CONTINUOUS}}$

2.8 Implicit Differentiation

If $x^4y + (x+y)^2 = 2y+2$, then what is $\frac{dy}{dx}$ at $(x,y) = (1,0)$?

$d(x^4y + (x+y)^2) = d(2y+2)$

$d(x^4)y + x^4dy + 2(x+y)d(x+y) = 2dy + 0$

$(4x^3dx)y + x^4dy + 2(x+y)(dx+dy) = 2dy$

At $(1,0)$: $\underline{4 \cdot 1^3 dx \cdot 0} + \underline{1^4 dy} + \underline{2(1+0)(dx+dy)} = 2dy$

$dy + 2dx + \cancel{2dy} = \cancel{2dy}$

$dy + 2dx = 0 \Rightarrow dy = -2dx \Rightarrow \boxed{\frac{dy}{dx} = -2}$

2.5 Derivatives

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(e^x)' = e^x$

$(a^x) = a^x \ln a \qquad a^x = (e^{(\ln a)})^x = e^{((\ln a)\cdot x)}$

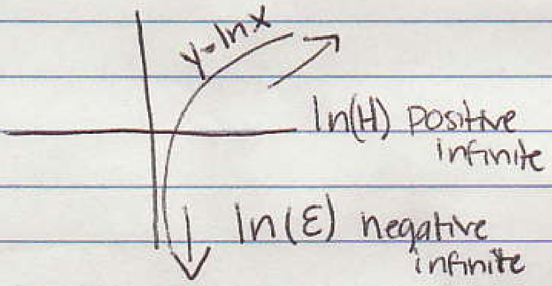
a positive constant

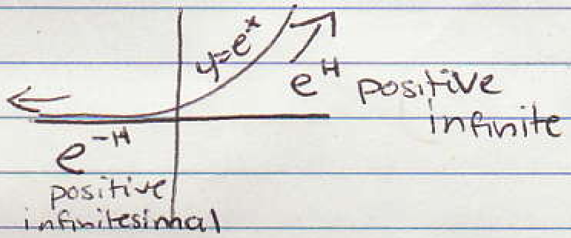
$(\log_a x)' = 1/(x \ln a) \qquad \log_a x = \dfrac{\ln x}{\ln a} = \left(\dfrac{1}{\ln a}\right)\ln x$

$(\ln x)' = 1/x$

3.3+ Limits

Let $0 < \varepsilon \approx 0$ and H positive infinite





$$\lim_{x \to 1^+} \frac{\ln(x-1) + x}{5 + \log_3(x-1)}$$

$= st\,(f(1+\varepsilon))$

$$f(1+\varepsilon) = \frac{\ln(1+\varepsilon-1) + (1+\varepsilon)}{5 + \log_3(1+\varepsilon-1)}$$

$$= \frac{\ln\varepsilon + 1 + \varepsilon}{5 + \log_3 \varepsilon}$$

$$= \frac{\ln\varepsilon + 1 + \varepsilon}{5 + \frac{1}{\ln 3}\cdot \ln\varepsilon}$$

Write $-K = \ln\varepsilon$

$$\frac{-K + 1 + \varepsilon}{5 + \frac{1}{\ln 3}(-K)}$$

$$st\left(\frac{-K+1+\varepsilon}{5 - K/\ln 3}\right) = st\left(\frac{(-K+1+\varepsilon)/K}{(5-K/\ln 3)/K}\right)$$

$$= st\left(\frac{-1 + 1/K + \varepsilon/K}{5/K - 1/\ln 3}\right)$$

$$\frac{-1 + 0 + 0}{0 - 1/\ln 3} = \boxed{\ln 3}$$

2.4 Inverse function rule:

If $y=f(x)$ & $\underbrace{x=g(y)}$:

$$\frac{dx}{g'(y)\,dx}=\frac{g'(y)\,dy}{g'(y)\,dx}$$

$$\boxed{\frac{1}{g'(y)}=\frac{dy}{dx}=f'(x)}$$

2.5 (Trig + exponential + log) $\boxed{a \text{ constant} > 0}$

$f(x)$	$\sin x$	$\cos x$	e^x	$\ln x$	a^x	$\log_a x$
$f'(x)$	$\cos x$	$-\sin x$	e^x	$1/x$	$a^x \ln a$	$1/(x \ln a)$
$f(x)$	$\tan x$	$\cot x$	$\sec x$	$\csc x$		
$f'(x)$	$\sec^2 x$	$-\csc^2 x$	$\sec x \tan x$	$-\csc x \cot x$		

2.6 (Chain rule)

If $y=f(u)$ & $u=g(x)$:

$$[f(g(x))]'=\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=f'(u)\,g'(x)$$

$$=f'(g(x))\,g'(x)$$

If $x=f(t)$ & $y=g(t)$:

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{g'(t)}{f'(t)}$$

(Higher Derivatives)

2.7 $f''(x) = (f'(x))'$
$f^{(3)}(x) = (f''(x))'$
$f^{(4)}(x) = (f^{(3)}(x))'$
$\vdots$

If $y = f(x)$:

$\dfrac{d^2y}{dx^2} = f''(x)$ $\quad d^2y = f''(x)(dx)^2$

$\dfrac{d^3y}{dx^3} = f^{(3)}(x)$ $\quad d^3y = f^{(3)}(x)(dx)^3$

$\vdots$

2.8 (Implicit Differentiation)

Example: $x^4y + (x+y)^2 = 2y+3$

$d(x^4y + (x+y)^2) = d(2y+3)$

$d(x^4)y + x^4dy + 2(x+y)d(x+y) = 2dy + 0$

$(4x^3dx)y + x^4dy + 2(x+y)(dx+dy) = 2dy$

$[4x^3y + 2(x+y)]dx + [x^4 + 2(x+y)]dy = 2dy$

$$\frac{[4x^3y + 2(x+y)]dx}{[2-x^4-2(x+y)]dx} = \frac{[2-x^4-2(x+y)]dy}{[2-x^4-2(x+y)]dx}$$

$$\boxed{\frac{4x^3y+2(x+y)}{2-x^4-2(x+y)} = \frac{dy}{dx}}$$

~~A~~ ~~[illegible]~~ 3.2 (Related rates)

Example: A spherical balloon is losing volume, ~~[illegible]~~ currently at a rate of **2** cm^3/second. If the radius is currently 25 cm, what is the rate of change of the radius right now?

$$V = \frac{4}{3}\pi r^3 \Rightarrow dV = \frac{4}{3}\pi\, d(r^3)$$

$$\Rightarrow dV = \frac{4}{3}\pi \cdot 3r^2\, dr = 4\pi r^2\, dr$$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

Right now, $r = 25\,cm$ & $dV/dt = -2\,cm^3/s$

so $\frac{dr}{dt} = \frac{1}{4\pi(25\,cm)^2}(-2\,cm^3/s) = \boxed{\frac{-1\,cm/s}{1250}}$

$$= \boxed{-0.0008\,cm/s}$$

3.3 Limits.

$$\lim_{x \to c} f(x) = \underbrace{st(f(c+\varepsilon))}$$

Must exist & be same for <u>all</u> nonzero infinitesimal ε

$$\lim_{x \to c^+} f(x) = \underbrace{st(f(c+\varepsilon))}$$

Must exist & be same for all positive infinitesimal ε

$$\lim_{x \to c^-} f(x) = \underbrace{st(f(c-\varepsilon))}$$

Must exist & be same for all positive infinitesimal ε.

☆ See sections 1.5 & 1.6 to review computation of $st(\ldots)$.

not in book

☆ Additional rules for $\ln x$ & e^x:

$0 < \varepsilon \approx 0 \Rightarrow \ln \varepsilon$ negative infinite

$0 < H$ infinite $\Rightarrow \begin{cases} e^{-H} & \text{positive infinitesimal} \\ e^{H} & \text{positive infinite} \\ \ln H & \text{positive infinite.} \end{cases}$

3.4 Continuity

$f(x)$ is cts. at $c \iff \lim\limits_{x \to c} f(x) = f(c)$

Right hand continuity: $\lim_{x \to c^+} f(x) = f(c)$
Left hand continuity: $\lim_{x \to c^-} f(x) = f(c)$

$f(x)$ is cts. on $[a,b]$

$$\iff \begin{cases} \lim\limits_{x \to c} f(x) = f(c) \text{ for all } c \text{ in } (a,b) \\ \lim\limits_{x \to a^+} f(x) = f(a) \\ \lim\limits_{x \to b^-} f(x) = f(b) \end{cases}$$

$\iff$ $f(x)$ avoids division by 0, $\ln(0)$, $\ln(\text{negative})$, $\sqrt[\text{even}]{\text{negative}}$

for "elementary" formulas

3.8

Theorem	Assumes f cts. on $[a,b]$	Assumes f' exists on (a,b)	Assumes...	Concludes some c in (a,b) has
IVT	✓		$f(a)<0<f(b)$ or $f(a)>0>f(b)$	$f(c)=0$
EVT	✓			$f(c) = \max\ldots$
RT	✓	✓	$f(a)=f(b)=0$	$f'(c)=0$
MVT	✓	✓		$f'(c)=\frac{f(b)-f(a)}{b-a}$