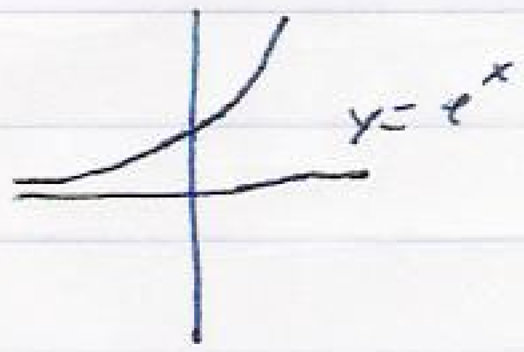
Notes 11-09-11

Today: horizontal & vertical asymptotes (5.3)

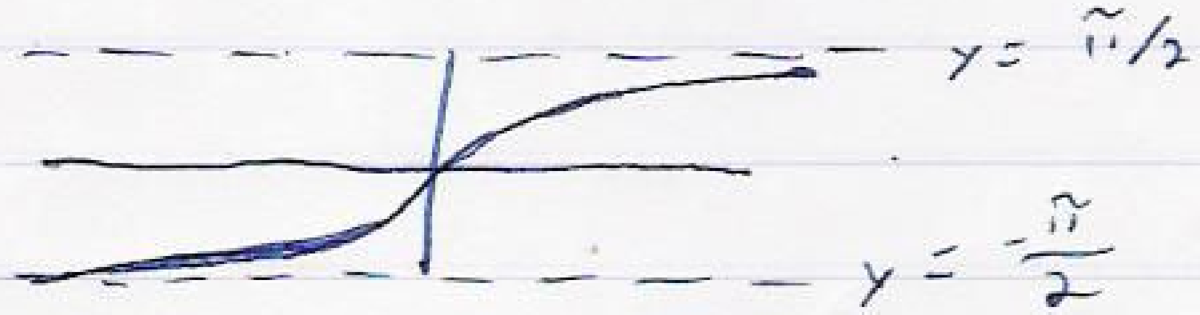
"$y = f(x)$ has a horizontal asymptote (HA) $y = L$" = or means $\lim\limits_{x \to \infty} f(x) = L$ or $\lim\limits_{x \to \infty} f(x) = L$.

$y = e^x$ has HA $y = 0$: $\lim\limits_{x \to -\infty} e^x = 0$



$y = 0$

$y = \arctan(x)$ has 2 HA's: $y = \frac{\pi}{2}$ & $-\frac{\pi}{2}$



$\lim\limits_{x \to \infty} \arctan(x) = \frac{\pi}{2}$ $\lim\limits_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}$

Recall $y = \arctan(x) \iff \begin{cases} x = \tan(y) \\ -\frac{\pi}{2} < y < \frac{\pi}{2} \end{cases}$

Find the HA's, if any, of:

$$y = f(x) = \frac{x^2 - 7x + 3}{(x-3)(2x+5)}$$

$\lim_{x \to \infty} f(x) = ?$

Plug in $x = H$

where H is positive infinite

$\lim_{x \to -\infty} f(x) = ?$

plug in $x = -H$

where H is positive infinite.

SO: $$\frac{H^2 - 7H + 3}{(H-3)(2H+5)} = \frac{(H^2 - 7H + 3)/H^2}{(H-3)(2H+5)/H^2} =$$

$$= \frac{(H^2 - 7H + 3)/H^2}{[(H-3)/H][(2H+5)/H]} = \frac{1 - 7/H + 3/H^2}{(1-3/H)(2+5/H)} =$$

$$\simeq \frac{1 - 0 + 0}{(1-0)(2+0)} = \frac{1}{2}$$

$\lim_{x \to \infty} f(x) = \frac{1}{2}$

$\lim_{x \to -\infty} f(x) = \frac{1}{2}$

$$y = \frac{x^3+5}{3-4x^3} \qquad HA: \ y = -\frac{1}{4}$$

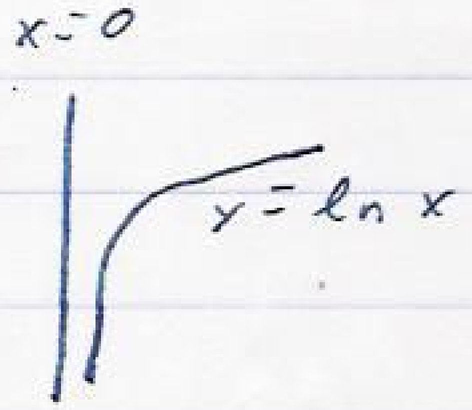
$x = H$: $\quad y = \dfrac{H^3+5}{3-4H^3} = \dfrac{(H^3+5)/H^3}{(3-4H^3)/H^3}$

$x = -H$

$$= \frac{1+5/H^3}{3/H^3-4} \approx \frac{1+0}{0-4} = -\frac{1}{4}$$

"$y = f(x)$ has a vertical asymptote (VA) $x = c$" means:

$$\lim_{x \to c^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to c^-} f(x) = \pm\infty$$



$$\lim_{x \to 0^+} \ln x = -\infty$$

Example: $y = \dfrac{x^3+5}{3-4x^3}$

Look for discontinuity, Only possibility $3-4x^3=0$

$3-4x^3 = 0 \to 3 = 4x^3 \to \frac{3}{4} = x^3 \to \sqrt[3]{\frac{3}{4}} = x$

Try $x = \sqrt[3]{\frac{3}{4}} + \varepsilon$: $\quad y = \dfrac{\left(\sqrt[3]{\frac{3}{4}}+\varepsilon\right)^3+5}{3-4\left(\sqrt[3]{\frac{3}{4}}+\varepsilon\right)^3} \quad \begin{matrix} \approx \frac{3}{4}+5 > 0 \\ \approx 0 \end{matrix}$

$0 \neq \varepsilon \approx 0$

$y = \dfrac{\text{positive, non-infinitesimal}}{\text{infinitesmal}}$ $\qquad$ y is infinite

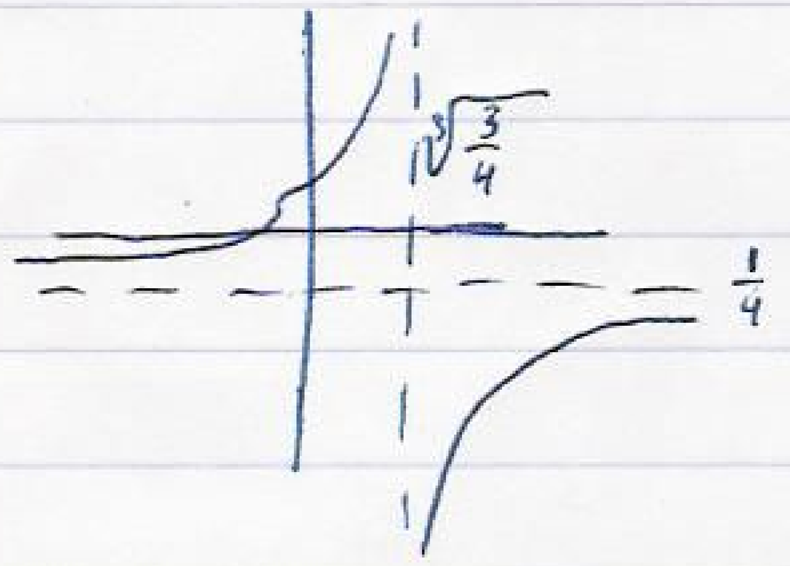
A little more algebra shows that

$$\begin{cases} \varepsilon > 0 \Rightarrow & y \text{ negative infinite} \\ \varepsilon < 0 \Rightarrow & y \text{ Positive infinite} \end{cases}$$

$$\lim_{x \to \sqrt[3]{\frac{3}{4}}^{+}} \frac{x^3+5}{3-4x^3} = -\infty$$

VA: $x = \sqrt[3]{\frac{3}{4}}$

$$\lim_{x \to \sqrt[3]{\frac{3}{4}}} \frac{x^3+5}{3-4x^3} = \infty$$



$$y = \frac{(3x-3)(x+2)}{x-1}$$

is not cont. at $x=1$,

but when $x \neq 1$ $y = 3(x+2)$

so no VA