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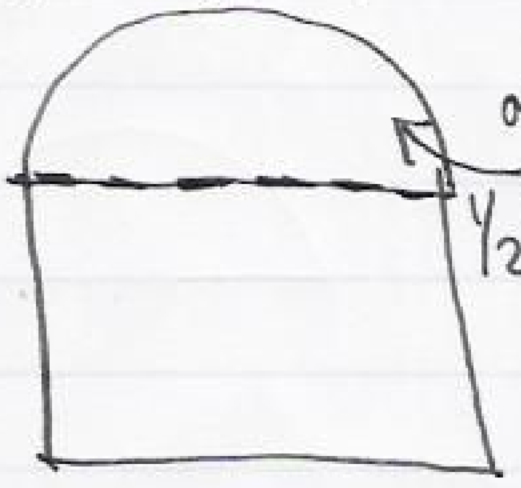
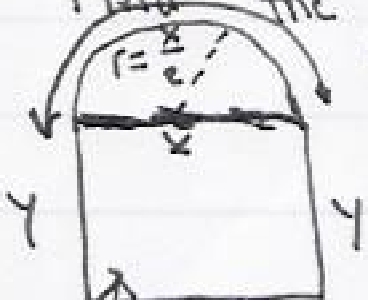
Minimize the length of fencing needed to enclose a rectangular region adjoined with a semicircle region with total area 500 ft^2.
Find the least possible length of fence.

1)



area= $\frac{1}{2}(\pi r^2)$



$\pi r = \frac{\pi x}{2}$

2) Minimize: $L = 2y + x + \frac{\pi x}{2}$

area=xy

3) Constraint: $500\text{ft}^2 = A = xy$

area of semicircle $= \frac{\pi x^2}{8}$.

$\frac{1}{2}(\pi r^2) = \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = \frac{1}{2}\pi\left(\frac{x^2}{4}\right) = \boxed{\frac{\pi x^2}{8}}$

4) ~~Constraint~~ $A - \frac{\pi x^2}{8} = xy \Rightarrow \frac{A}{x} - \frac{\pi x}{8} = y$

$$L = 2\underbrace{\left(\frac{A}{x} - \frac{\pi x}{8}\right)}_{y} + x + \frac{\pi x}{2}$$

5) $L = \frac{2A}{x} - \frac{\pi x}{4} + x + \frac{\pi x}{2}$

$L = \frac{2A}{x} + x + \frac{\pi x}{4}$

$L = 2Ax^{-1} + x\left(1 + \frac{\pi}{4}\right)$

$\frac{dL}{dx} = 2A(-1)x^{-1-1} + (1)\left(1 + \frac{\pi}{4}\right)$

$\frac{dL}{dx} = -2Ax^{-2} + 1 + \frac{\pi}{4}$

$\frac{dL}{dx} = 1 + \frac{\pi}{4} - \frac{2A}{x^2}$

Critical Points: where derivative is 0 or undefined

$\frac{dL}{dx}$ is undefined at 0, but we should restrict x to $(0,\infty)$ anyway: x must be positive because its a length. Ignore $x=0$

Solve $\frac{dL}{dx} = 0$

$0 = 1 + \frac{\pi}{4} - \frac{2A}{x^2}$

$\frac{2A}{x^2} = 1 + \frac{\pi}{4} = \frac{4+\pi}{4}$

$\frac{2A}{\cancel{x^2}} \cdot 4\cancel{x^2} = \frac{4+\pi}{\cancel{4}} \cdot \cancel{4}x^2$

$8A = (4+\pi)x^2$

$\frac{8A}{4+\pi} = x^2$

$\pm\sqrt{\frac{8A}{4+\pi}} = x$ Ignore the negative solution

Just 1 critical point in $(0,\infty)$:

$x=\sqrt{\frac{8A}{4+\pi}}$, so this is probably optimal

You can check this using Direct Test or 2nd Derivative Test.

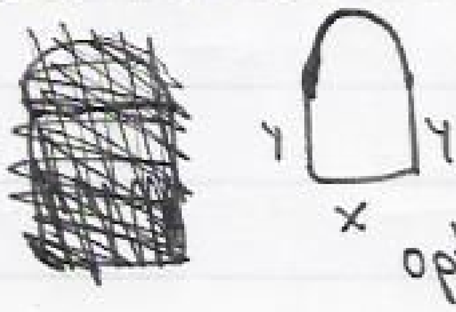
<u>Question asked for minimum L</u>

$$y = \frac{A}{x} - \frac{\pi x}{8} = \frac{A}{\sqrt{\frac{8A}{4+\pi}}} - \frac{\pi\sqrt{\frac{8A}{4+\pi}}}{8}$$

$$L = 2y + x + \frac{\pi x}{2}$$

$$L = 2\left(\left(\frac{A}{\sqrt{\frac{8A}{4+\pi}}}\right) - \frac{\pi\sqrt{\frac{8A}{4+\pi}}}{8}\right) + \sqrt{\frac{8A}{4+\pi}} + \frac{\pi}{2}\sqrt{\frac{8A}{4+\pi}}$$

Where $A = 500\ ft^2$



optimal?

$x=\sqrt{\frac{8A}{4+\pi}} = 23.66\ldots$

$y = \frac{500}{x} - \frac{\pi x}{8} = 11.83$

$L = 84.50\ldots.$ (feet)

<u>Related Rates</u>

A spherical tank is being filled with H_2O at a constant rate of $50\ cm^3/s$. If the tank has a radius $r=5m$ ($500cm$), how fast is water level rising when the tank is half full?

h = height of the water above bottom of tank

$\frac{dv}{dt} = 50\ cm^3/s$ V= volume of water in tank

V= sum of volumes of the many thin horizontal disks.

thickness dh disk has cross sectional area

$$V = \int_0^h A(h)\,dh \qquad \frac{d}{dh}\int_0^h A(h)\,dh$$

$$50\,cm^3/s = \frac{dV}{dt} = \frac{dV}{dh}\,\frac{dh}{dt} = A(h)\cdot\frac{dh}{dt}$$

$$50\,cm^3/s = \frac{dr}{dt} = \pi r^2 \cdot \frac{dh}{dt} = \frac{50\,cm^3/s}{\pi r^2} = \frac{dh}{dt}$$

When it is half full:  $= A(h) = \pi r^2$

$$\frac{50\,cm^3/s}{\pi(500\,cm)^2} = \boxed{6.36\cdot 10^{-5}\ cm/s}$$