Last time: Simpson's Rule:

Estimate $\int_a^b f(x)\,dx$ with

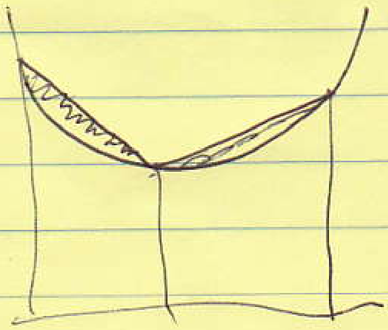
$$\frac{\Delta x}{3} \sum_{k=1}^{N/2 \leftarrow (N \text{ must be even})} \left(f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})\right)$$

where $\Delta x = \frac{b-a}{N}$, $x_k = a + k\Delta x$

Before that: Trapezoid Rule:

$$\frac{\Delta x}{2} \sum_{k=1}^{N} \left(f(x_{k-1}) + f(x_k)\right)$$

Error bounds: Guarantee a level of accuracy.



CU $\longleftrightarrow$ $f'' > 0$

Trapezoid rule overestimates

If you know that for all x in $[a,b]$, $-M \le f''(x) \le M$, then you can conclude that the error E_T for the trapezoid rule estimate of $\int_a^b f(x)\,dx$ is between $\dfrac{-M(b-a)^3}{12N^2}$ and $\dfrac{+M(b-a)^3}{12N^2}$

Find an M... Options

(1) Graph $f''(x)$ over $[a,b]$

(2) Use some algebra

(3) Use calculus to find the best possible M.

Example of algebra estimate for M:

$$\ln(3) = \int_1^3 \frac{1}{x}\,dx \qquad N = 15$$

Trapezoid Rule

$$f(x) = \frac{1}{x} = x^{-1} ; [a,b] = [1,3]$$

$$f'(x) = (-1)x^{-1-1} = (-1)x^{-2} = \boxed{-x^{-2}}$$

$$f''(x) = -(-2)x^{-2-1} = 2x^{-3} = \frac{2}{x^3}$$

$$1 \le x \le 3 \Rightarrow \frac{1}{1} \ge \frac{1}{x} \ge \frac{1}{3}$$

$$\Rightarrow \left(\frac{1}{1}\right)^3 \ge \left(\frac{1}{x}\right)^3 \ge \left(\frac{1}{3}\right)^3$$

$$1 \ge \frac{1}{x^3} \ge \frac{1}{27}$$

$$2 \ge \frac{2}{x^3} \ge \frac{2}{27} > -2$$

Use $M = 2$.

error at worst $\pm \dfrac{2(3-1)^3}{12(15)^2}$

$$= \pm 0.002963$$

$$\int_1^3 \frac{1}{x}\,dx:$$

E_T at worst $\pm \dfrac{2(3-1)^3}{12N^2}$

HW#1 How big must N be to guarantee the error is at worst ± 0.00001?

If you know that for all x in $[a,b]$, $-M \le f^{(4)}(x) \le M$, then you can conclude that the error E_S for Simpson's Rule's estimate of $\int_a^b f(x)\,dx$ is between

$$\frac{-M(b-a)^5}{180N^4} \quad \text{and} \quad +\frac{M(b-a)^5}{180N^4}$$

Apply Simpson's Rule to $\int_1^3 \frac{1}{x}\,dx$. $\qquad f(x) = \frac{1}{x}$

$$f(x) = \frac{1}{x} \quad \ldots \quad f''(x) = 2x^{-3}$$

$$f^{(3)}(x) = 2(-3)x^{-3-1} = -6x^{-4}$$

$$f^{(4)}(x) = -6(-4)x^{-4-1} = 24/x^5$$

$$1 \leq x \leq 3$$

$$1 \geq \frac{1}{x} \geq \frac{1}{3}$$

$$1 \geq \frac{1}{x^5} \geq \frac{1}{3^5} = \frac{1}{243}$$

$$24 \geq \frac{24}{x^5} \geq \frac{24}{243} > -24$$

$$M = 24 \Rightarrow E_s : \frac{\pm 24(3-1)^5}{180 N^4}$$

(at worst)

Say we want E_s at worst

$$\pm \underbrace{0.000001}_{10^{-6}}$$

What should N be?

$$10^{-6} = \frac{24(3-1)^5}{180 N^4}$$

$$\frac{180 \times 10^{-6}}{24(3-1)^5} = \left(\frac{1}{N^4}\right) \cancel{\Rightarrow \sqrt[4]{\frac{180 \times 10^{-6}}{24(3-1)^5}} \geq N}$$

$$\frac{24(3-1)^5}{180\times10^{-6}} = N^4$$

$$\sqrt[4]{\frac{24\times32}{180\times10^{-6}}} = N$$

$$45.4 = N$$

Round up to nearest even # for Simpson's Rule.

$$N = 46$$

So, $N=46$ & Simpson's Rule on $\int_1^3 \frac{1}{x}\,dx$ has an error at worst $\pm 10^{-6}$.

HW #2 Express $\ln(7)$ as an integral. Use Simpson's Rule with $N=8$ and prove an error bound (error guarantee).

Next, Find N with guaranteed error at worst $\pm 10^{-6}$

$$\int_1^2 \underbrace{e^{-x^2/2}}_{f(x)}\,dx$$

8 Trapezoid Rule

$N = 11$

Get an error bound.

$$f'(x) = e^{-x^2/2}\left(-x^2/2\right)'$$
$$= e^{-x^2/2}\left(-2x/2\right)$$
$$= -xe^{-x^2/2}$$

$$f''(x) = (-x)'e^{-x^2/2} + -x\underbrace{\left(e^{-x^2/2}\right)'}_{f'(x)}$$

$$= -1e^{-x^2/2} + -x\left(-xe^{-x^2/2}\right)$$

$$= \underbrace{(x^2-1)e^{-x^2/2}}$$

$M = 1$ works

used graph to find M

is between -1 & 1 when $1 \le x \le 2$

$$E_T \text{ at worst } \frac{\pm M(b-a)^3}{12N^2} \quad b=2,\ a=1,\ N=11$$

$= \pm 0.007575\ldots$