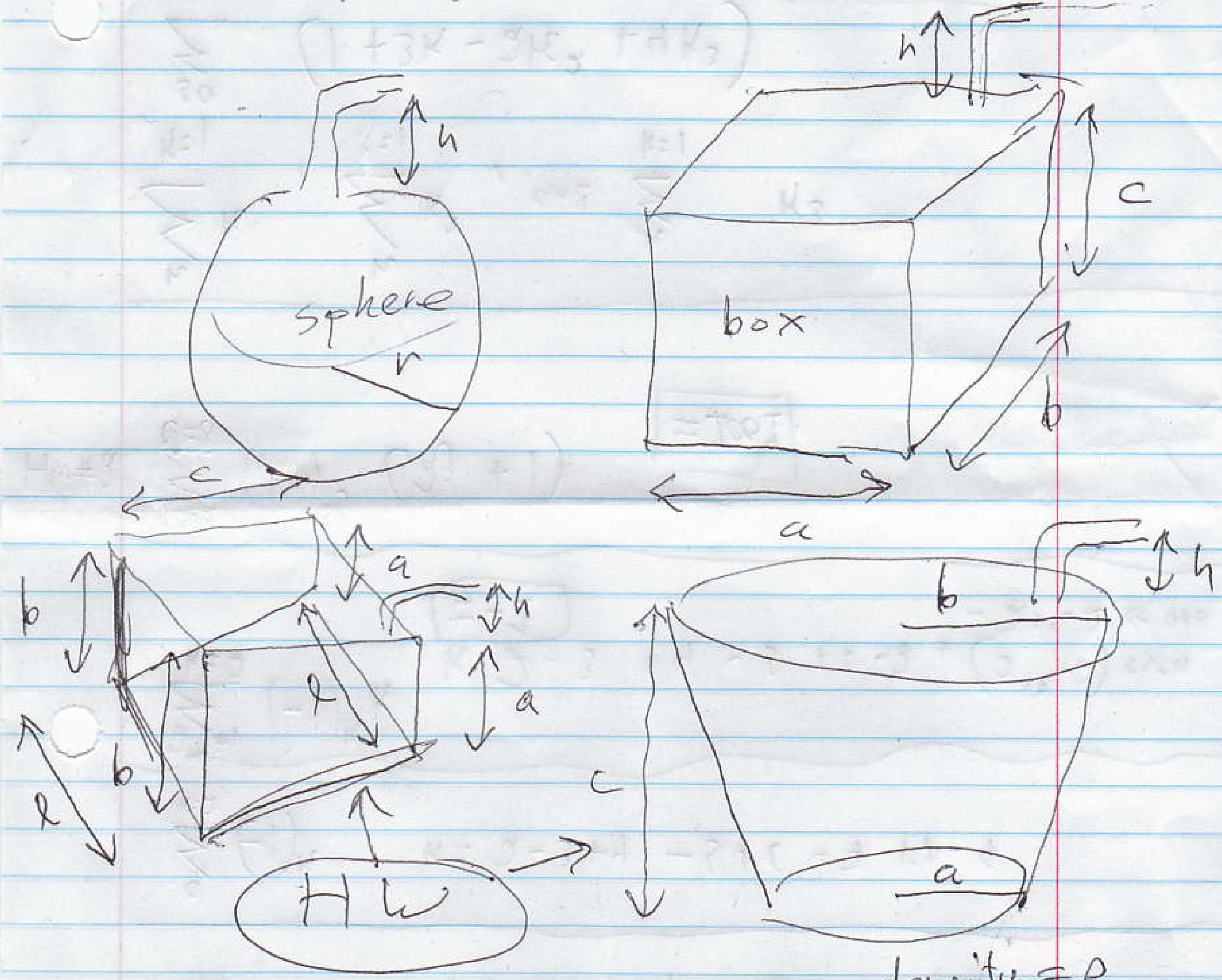
Emptying tanks



density $= \rho$

Start with a box:

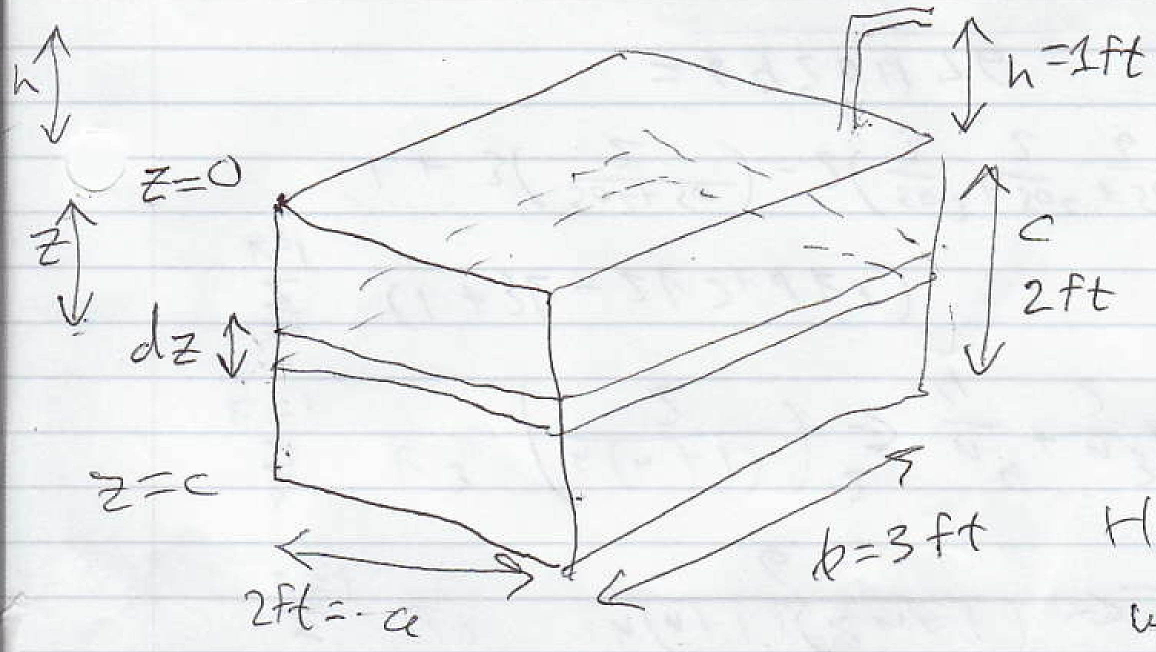
$\rho =$ density of liquid $= \dfrac{\text{weight or mass}}{\text{volume}}$

(if H_2O: $\boxed{62.4 \text{ lbs}/ft^3}$ ★

$10^3 \text{ kg}/m^3$

$9.8 \times 10^3 \text{ N}/m^3$

We'll use weight density



Before: tank full

After: tank empty

How much work done?

weight of slice = volume × density

$ab\,dz \qquad \rho$

slice raised $\quad z+h$

dW = Work done on slice = weight (force) × distance up/down

$$ab\rho\,dz\,(z+h)$$

$$\text{Work} = \int_{z=0}^{z=c} dW = \int_0^c \underbrace{ab\rho}_{\text{constants}}\,dz\,(z+h)$$

$$= \left(\int_0^c (z+\underbrace{h}_{\text{const.}})\,dz\right) ab\rho = \left(\frac{z^2}{2}+hz\right)\Big|_0^c ab\rho$$

$$= \left(\frac{c^2}{2}+hc-\left(\frac{0^2}{2}+h\cdot 0\right)\right) ab\rho$$

$$= \left(\frac{c^2}{2}+hc\right) ab\rho$$

E.g. $a = 2ft \quad b = 3ft \quad c = 2ft$

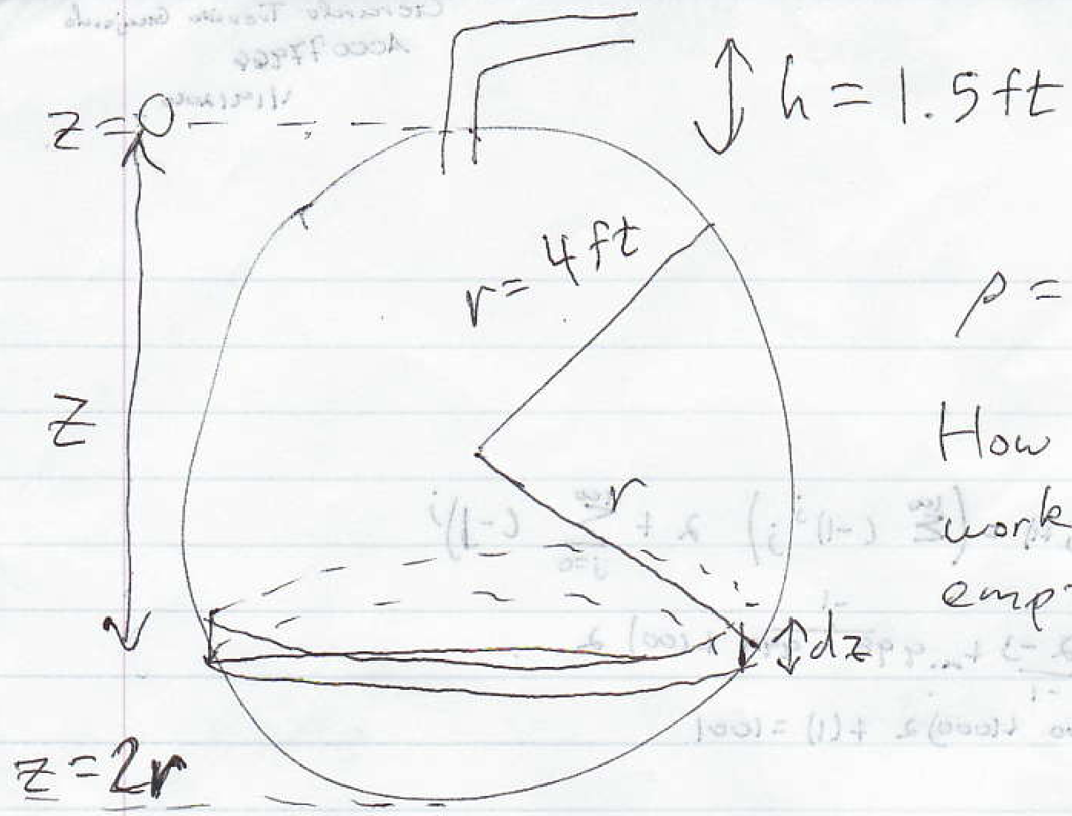
$h = 1ft \quad \rho = 62.4\ lb/ft^3$

$$W = \left(\frac{(2ft)^2}{2} + (1ft)(2ft)\right)(2ft)(3ft)\frac{62.4\ lb}{ft^3}$$

$$\underbrace{\left(\frac{2^2}{2} + 1\cdot 2\right)}_{4} ft^2 \cdot 2\cdot 3 ft^2 \cdot \frac{62.4\ lb}{ft^3}$$

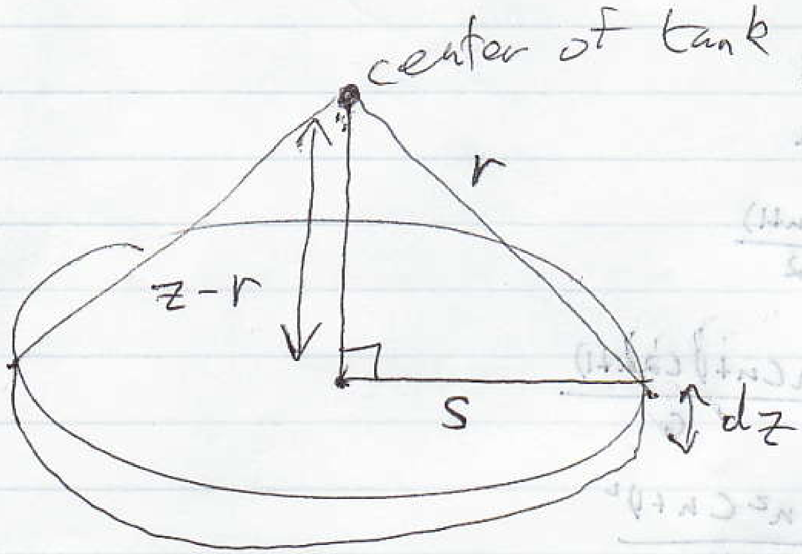
$$4 \cdot 2 \cdot 3 \cdot 62.4\ ft\ lb$$

$$W = 1497.6\ ft\ lb \approx 1500\ ft\ lbs$$



$\rho = 62.4\ lb/ft^3$

How much work to empty tank?



$$(z-r)^2 + s^2 = r^2$$

Volume of slice $= \underbrace{\pi s^2\, dz}_{\text{circular cylinder}}$

$\text{volume} = \pi\left(r^2 - (z-r)^2\right) dz$

weight = density × volume $= \rho\pi\left(r^2-(z-r)^2\right)dz$

$dW = \text{weight} \times (\text{distance lifted})$

$$\rho\pi\left(r^2-(z-r)^2\right)dz\,(z+h)$$

$$W = \int_{z=0}^{z=2r} dW = \int_0^{2r} \rho\pi\underbrace{\left(r^2-(z-r)^2\right)}_{r^2-(z^2-2zr+r^2)}dz(z+h)$$

$$= \rho\pi\int_0^{2r}\left(-z^2+2zr\right)(z+h)\,dz$$

$$= \rho\pi\int_0^{2r}\left(-z^3-z^2h+2z^2r+2zrh\right)dz$$

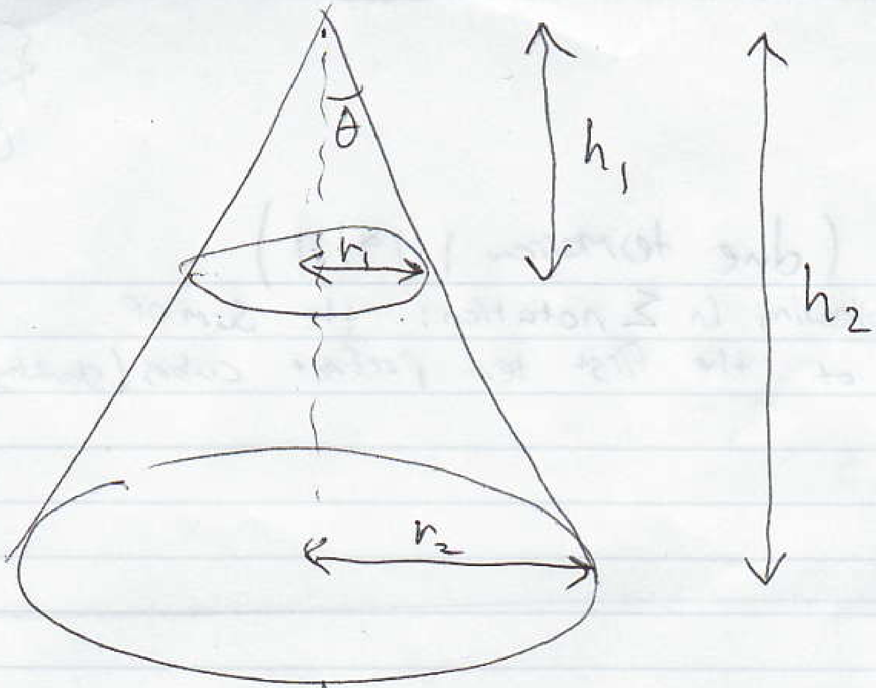
$$= \rho\pi\left(-\frac{z^4}{4}-\frac{z^3h}{3}+\frac{2rz^3}{3}+z^2rh\right)\Big|_0^{2r}$$

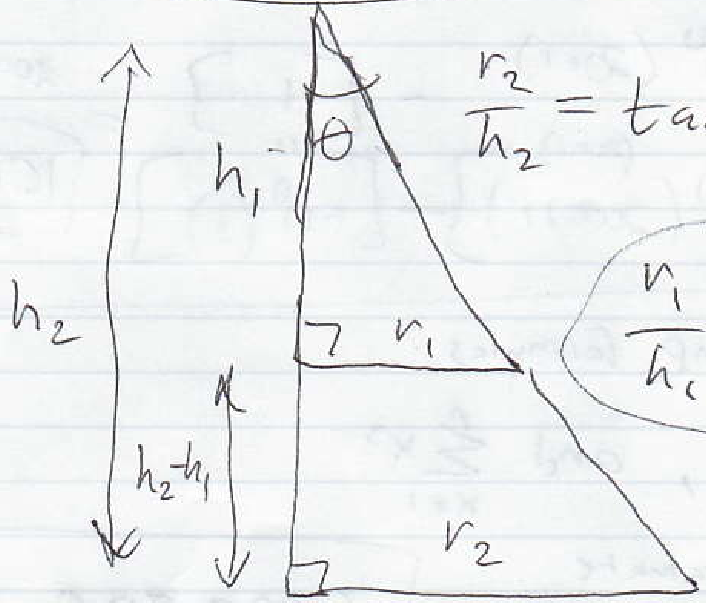
$$= \rho\pi\left(-\frac{16r^4}{4}-\frac{8r^3h}{3}+\frac{16r^4}{3}+4r^3h\right)$$

$(2r)^4 = 16r^4$

$$= \rho\pi\left(\frac{4}{3}r^4+\frac{4}{3}r^3h\right) = \rho\pi\left(\frac{4}{3}r^3\right)(r+h)$$

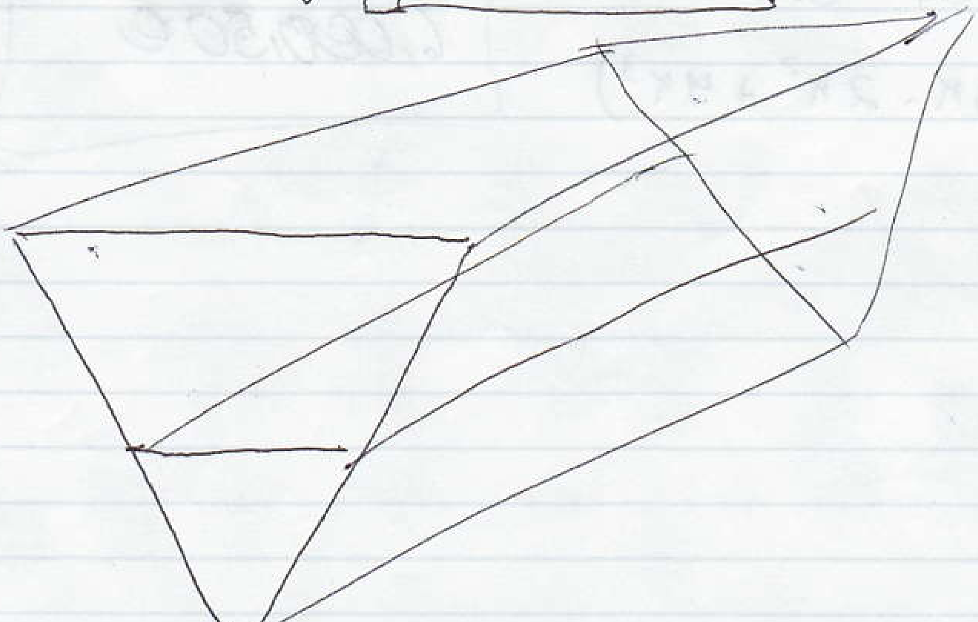
$$= \underbrace{\rho\underbrace{\left(\frac{4}{3}\pi r^3\right)}_{\text{volume of tank}}}_{\text{weight of } H_2O}\underbrace{(r+h)}_{\text{distance lifted from center}}$$

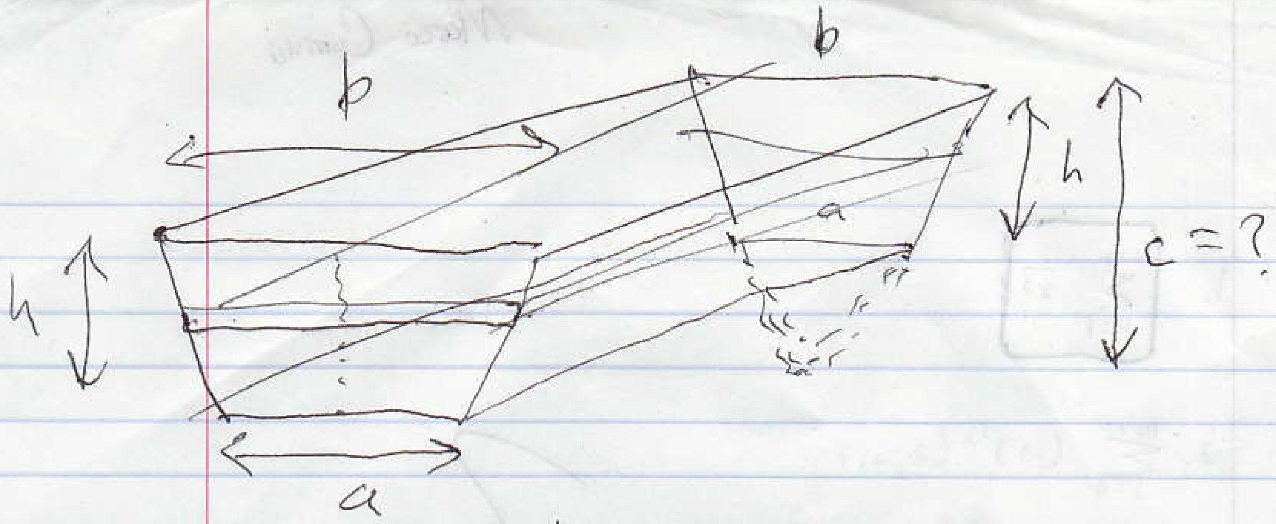


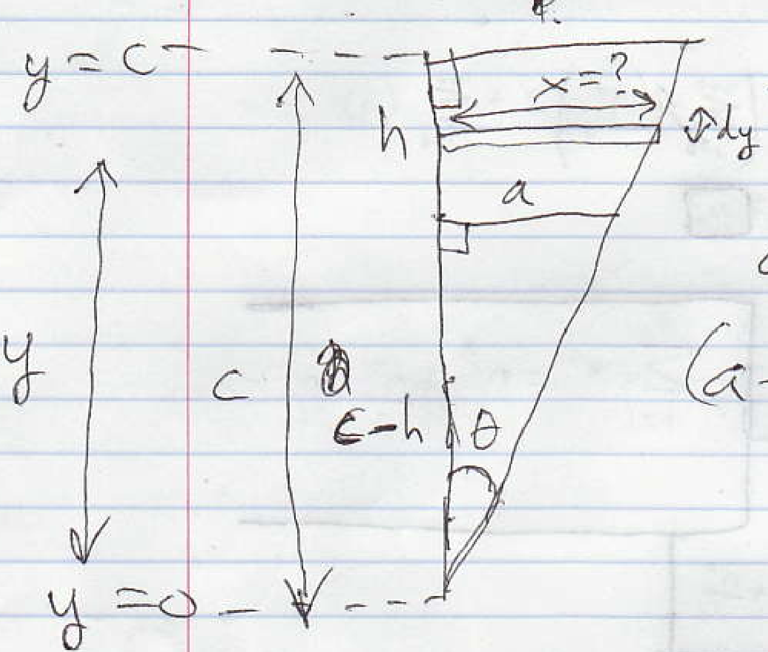


$$\frac{r_2}{h_2} = \tan\theta = \frac{r_1}{h_1}$$

$$\frac{r_1}{h_1} = \frac{r_2}{h_2}$$







$$\tan\theta = \frac{a}{c-h} = \frac{b}{c}$$

$$ac = b(c-h)$$

$$(a-b)c = -bh$$

$$c = \frac{-bh}{a-b}$$

$$c = \frac{bh}{b-a} \quad ☆$$

$$\frac{b}{c} = \tan\theta = \frac{x}{y} \implies x = \frac{by}{c} \quad ☆$$