Test 5 on Monday

Partial Fractions (8.8)

Separable Differential Equations (14.1)

Integration by Parts (7.4)

Linear Differential Equations (14.2–14.3)

Go to those sections of the book for practice problems...

Notes? OK.

Calculator? not OK.

$$\frac{dx}{dt} + p'(t)\,x = q(t)$$

E.g. $\dfrac{dx}{dt} + \underbrace{\dfrac{3}{2-t}}_{\frac{dp}{dt}}\, x = \underbrace{t}_{q}$ ☆

same: $(2-t)\dfrac{dx}{dt} + 3x = 2t - t^2$

$$\frac{dx}{dt} + \frac{dp}{dt} x = q$$

$$e^p \left(\frac{dx}{dt} + \frac{dp}{dt} x \right) = e^p q$$

$$d(e^p x) = d(e^p) x + e^p dx$$

$$\frac{d(e^p x)}{dt} = \frac{(e^p dp) x + e^p dx}{dt}$$

$$\frac{d}{dt}(e^p x) = \underbrace{e^p \left(\frac{dp}{dt} x + \frac{dx}{dt} \right)}_{\text{same}}$$

$$\frac{d(e^p x)}{dt} = e^p q$$

$$\frac{dx}{dt} + \underbrace{\frac{3}{2-t}}_{\frac{dp}{dt}} x = t$$

$$p = \int \frac{3dt}{2-t} = \int \frac{3(-dw)}{w}$$

$$w = 2-t \qquad dw = -dt$$

$$-dw = dt$$

$$p = -3\int \frac{dw}{w} = -3\ln|w| \quad (\text{pick } c=0)$$

$$p = -3\ln|2-t| \qquad e^p = e^{-3\ln|2-t|}$$

$$e^p = \left(e^{\ln|2-t|}\right)^{-3} = |2-t|^{-3} = \frac{1}{|2-t|^3}$$

Multiply $\left(\frac{dx}{dt} + \underbrace{\frac{3}{2-t}}_{\frac{dp}{dt}} x = \underbrace{t}_{q}\right)$ by e^p

$$\frac{d(e^p x)}{dt} = e^p\left(\frac{dx}{dt} + x\frac{dp}{dt}\right) = e^p q$$

$$\frac{d(x/|2-t|^3)}{dt} = \frac{t}{|2-t|^3}$$

$$\int d(x/|2-t|^3) = \int \frac{t\,dt}{|2-t|^3}$$

$$\int d(x/(2-t)^3) = \int \frac{t\,dt}{(2-t)^3}$$

$$x/(2-t)^3 = \int \frac{t\,dt}{(2-t)^3}$$

$w = 2 - t \quad dw = -dt \quad dt = -dw \quad t = 2 - w$

$$\frac{x}{(2-t)^3} = \int \frac{(2-w)(-dw)}{w^3} = \int (w' - 2) w^{-3} \, dw$$

$$= \int (w^{-2} - 2w^{-3}) \, dw = \frac{w^{-2+1}}{-2+1} - 2\frac{w^{-3+1}}{-3+1} + c$$

$$= \frac{w^{-1}}{-1} - 2\frac{w^{-2}}{-2} + c = -\frac{1}{w} + \frac{1}{w^2} + c$$

$$= \frac{-1}{2-t} + \frac{1}{(2-t)^2} + c$$

$$x = -(2-t)^2 + (2-t) + c(2-t)^3$$

$$x^2 \frac{dx}{dt} - 3\cos t + 5 = t$$

$$x^2 \frac{dx}{dt} = t + 3\cos t - 5$$

$$\int x^2 \, dx = \int (t + 3\cos t - 5) \, dt$$

$$\frac{x^3}{3} = \frac{t^2}{2} + 3\sin t - 5t + c$$

$$x^3 = \frac{3t^2}{2} + 9\sin t - 15t + 3c$$

$$x = \sqrt[3]{\frac{3}{2}t^2 + 9\sin t - 15t + 3c}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

(circles centered about the origin)

$$dy = -\frac{x\,dx}{y}$$

$$\int y\,dy = \int -x\,dx$$

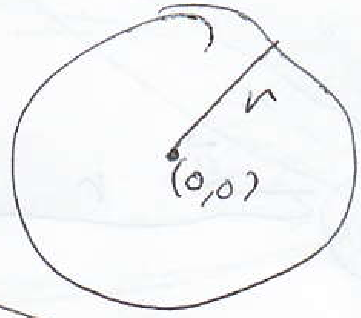
$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

$$x^2 + y^2 = 2c$$

If $\sqrt{2c} = r$, then

$$x^2 + y^2 = r^2$$



$$\frac{dz}{dx} = (1+z)(2-z)$$

$$dz = (1+z)(2-z)dx$$

$$\int \frac{dz}{(1+z)(2-z)} = \int dx = x + c$$

$$\frac{1}{(1+z)(2-z)} = \frac{A}{1+z} + \frac{B}{2-z}$$

$$1 = A(2-z) + B(1+z)$$

true for all z

$$2-z=0 \iff z=2$$

$$z=2 \Rightarrow 1 = A(0) + B(1+2) = 3B$$

$$1/3 = B$$

$$1+z=0 \iff z=-1$$

$$z=-1 \Rightarrow 1 = A(2-(-1)) + B(0) = 3A$$

$$1/3 = A$$

$$\int \left(\frac{1/3}{1+z} + \frac{1/3}{2-z} \right) dz = x + c$$

$$\frac{1}{3} \int \frac{dz}{1+z} + \frac{1}{3} \int \frac{dz}{2-z} = x + c$$

$u = 1+z$ $\qquad$ $v = 2-z$

$du = dz$ $\qquad$ $dv = -dz$

$-dv = dz$

$$\frac{1}{3} \int \frac{du}{u} + \frac{1}{3} \int \frac{-dv}{v} = x + c$$

$$\frac{1}{3} \ln|u| - \frac{1}{3} \ln|v| = x + c$$

$$\frac{1}{3}\ln|1+z| - \frac{1}{3}\ln|2-z| = x+c$$

↑ Implicit solution.

$$\frac{1}{3}\ln\left|\frac{1+z}{2-z}\right| = x+c \Rightarrow \frac{1+z}{2-z} = \underbrace{\pm e^{3(x+c)}}_{\underbrace{\pm e^{c}}_{A}\, e^{3x}}$$

$$\frac{1+z}{2-z} = Ae^{3x}$$

$$1+z = Ae^{3x}(2-z) = 2Ae^{3x} - zAe^{3x}$$

$$zAe^{3x} + z = -1 + 2Ae^{3x}$$

$$z\left(Ae^{3x}+1\right) = -1 + 2Ae^{3x}$$

$$z = \frac{-1+2Ae^{3x}}{Ae^{3x}+1}$$

IBP

$$\int \underbrace{x^2}_{u}\, \underbrace{\cos(x/5)\,dx}_{dv}$$

$$\int u\,dv = uv - \int v\,du$$

$du = 2x\,dx$

$v = \int \cos\frac{x}{5}\,dx$

$w = \frac{x}{5}$ $\quad dw = \frac{dx}{5}$ $\quad 5\,dw = dx$ $\quad v = \int \cos w\,(5\,dw)$

$v = 5\sin w$ (pick $c=0$) $v = 5\sin\frac{x}{5}$

$$\int \underset{u}{x^2}\, \underset{dv}{\cos\frac{x}{5}\,dx} = \underset{u}{x^2}\cdot \underset{v}{5\sin\frac{x}{5}} - \underbrace{\int \underset{v}{\left(5\sin\frac{x}{5}\right)}\underset{du}{2x\,dx}}_{\text{Use IBP}}$$

$$\int x^2\cos\frac{x}{5}\,dx = 5x^2\sin\frac{x}{5} - 10\boxed{\int \underset{u_2}{x}\,\underbrace{\sin\frac{x}{5}\,dx}_{dv_2}}^{\text{again}}$$

$du_2 = dx$

$v_2 = \int \sin\frac{x}{5}\,dx = \int \sin w\,(5\,dw) = -5\cos w$

pick $c=0$

$v_2 = -5\cos\frac{x}{5}$

$$\int \underset{u_2}{x}\,\underset{dv_2}{\sin\frac{x}{5}\,dx} = \underset{u_2}{x}\underset{v_2}{\left(-5\cos\frac{x}{5}\right)} - \int \underset{v_2}{\left(-5\cos\frac{x}{5}\right)}\underset{du_2}{dx}$$

$$\int x\sin\frac{x}{5}\,dx = -5x\cos\frac{x}{5} + 5\underbrace{\int \cos\frac{x}{5}\,dx}_{5\sin\frac{x}{5}+c}$$

$$\int x^2\cos\frac{x}{5}\,dx = \boxed{5x^2\sin\frac{x}{5} + 50x\cos\frac{x}{5} - 250\sin\frac{x}{5} + c}$$ ✓

$$\begin{array}{c|c}
\downarrow d & \downarrow \int \\
x^2 & \cos\frac{x}{5} \\
2x & 5\sin\frac{x}{5} \\
2 & -25\cos\frac{x}{5} \\
0 & -125\sin\frac{x}{5}
\end{array}$$

$\oplus : 5x^2 \sin\frac{x}{5}$

$\ominus : (50x)\cos\frac{x}{5}$

$\oplus : -250\sin\frac{x}{5}$

$$\int x^2 \cos\frac{x}{5}\,dx = 5x^2 \sin\frac{x}{5} + 50x\cos\frac{x}{5} - 250\sin\frac{x}{5} + c \quad \checkmark$$

Shortcut didn't work for

$$\int e^x \cos x\,dx,\ \int \sec^3 x\,dx, \ldots$$