Last time: improper integrals (6.7)
Today: sequences and series (9.1, 9.2)

Sequences: $s_n = \dfrac{(-1)^n}{n}$

$s_1 = -1, \quad s_2 = \frac{1}{2}, \quad s_3 = -\frac{1}{3}, \quad s_4 = \frac{1}{4}, \ \ldots$

$\lim_{n\to\infty} s_n = 0$, meaning that we can get s_n as close to 0 as we want, merely by making n sufficiently large.
(Equivalently, if H is an infinite positive hyperinteger then s_H is infinitely close to 0.)

Simple argument: if n is large,

$s_n = \dfrac{(-1)^n}{\text{large}} = \dfrac{\pm 1}{\text{large}} = \pm\,\text{small}$, so $\lim_{n\to\infty} s_n = 0$

$b_n = \dfrac{n^3+5}{(2n+1)^2(n+4)}$ HW #1 Find $\lim_{n\to\infty} b_n$.

You may want to review section 5.1

$a_n = n^2 \Rightarrow \lim_{n\to\infty} a_n = \infty$,

meaning that ...

... we can get a_n as large (and positive) as we wish merely by making n sufficiently large.

(Equivalently, if H is an infinite positive hyperinteger, then a_H is an infinite positive hyperreal.)

Simple argument: large^2 = large, so

$\lim_{n \to \infty} a_n = \infty$.

(Hint: l'Hospital's rule)

$c_m = \dfrac{e^m}{m}$ HW #2 Find $\lim_{m \to \infty} c_m$

$d_k = \left(\dfrac{\ln k}{k}\right)(-1)^k$ HW #3 Find $\lim_{k \to \infty} d_k$

Series: $\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$

$\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \left(\sum_{k=1}^{n} a_k \right)$ by definition

$a_k = \left(\dfrac{1}{k} - \dfrac{1}{k+1}\right) \Rightarrow \sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} \left(\dfrac{1}{k} - \dfrac{1}{k+1}\right)$

$= \lim_{n \to \infty} \left[\left(\dfrac{1}{1} - \cancel{\dfrac{1}{2}}\right) + \left(\cancel{\dfrac{1}{2}} - \cancel{\dfrac{1}{3}}\right) + \cdots + \left(\cancel{\dfrac{1}{n}} - \dfrac{1}{n+1}\right)\right]$

$= \lim_{n \to \infty} \left(1 - \dfrac{1}{n+1}\right) = 1 - 0 = 1$ (Called a telescoping series.)

Geometric series: If $a \neq 1$, then

$$\sum_{k=0}^{n} a^k = a^0 + a^1 + a^2 + \cdots + a^n$$

$$= (a^0 + a^1 + a^2 + \cdots + a^n)(1-a)/(1-a)$$

$$= \frac{(a^0 + a^1 + a^2 + \cdots + a^n - a^1 - a^2 - \cdots - a^{n+1})}{(1-a)}$$

$$= \frac{a^0 - a^{n+1}}{1-a} = \frac{1 - a^{n+1}}{1-a}$$

If $-1 < a < 1$, then

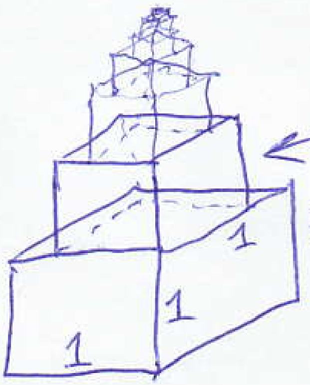
$$\sum_{k=0}^{\infty} a^k = \lim_{n\to\infty} \sum_{k=0}^{n} a^k = \lim_{n\to\infty} \frac{1-a^{n+1}}{1-a}$$

$$= \frac{1-0}{1-a} = \frac{1}{1-a}$$

because when $-1 < a < 1$, $a^{+\text{large}} = \pm$ small.

(Try $(-0.8)^{100}$ on your calculator.)

Consider an infinite stack of cubes, where each cube has half the volume of the one below it, except for the bottom which has volume 1 and no cube beneath it.



$\left(\frac{1}{\sqrt[3]{2}}\right)^3 = \frac{1}{2}$ (2nd cube)

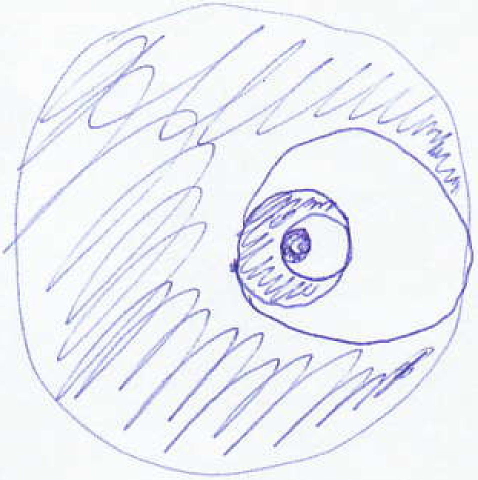
$1^3 = 1$ (bottom cube)

$$\text{volume} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$
$$= \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots$$
$$= \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2}$$

$$\text{height} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{8}} + \cdots$$
$$= \left(\frac{1}{\sqrt[3]{2}}\right)^0 + \left(\frac{1}{\sqrt[3]{2}}\right)^1 + \left(\frac{1}{\sqrt[3]{2}}\right)^2 + \left(\frac{1}{\sqrt[3]{2}}\right)^3 + \cdots$$
$$= \sum_{k=0}^{\infty}\left(\frac{1}{\sqrt[3]{2}}\right)^k = \frac{1}{1-\frac{1}{\sqrt[3]{2}}} = \frac{\sqrt[3]{2}}{\sqrt[3]{2}-1} \approx 4.85$$

HW #4 surface area = ?

(Don't count areas where cubes touch eachother.)



Starting with a disc of radius 1, remove a disc of radius $\frac{1}{2}$; add back a disc of radius $\frac{1}{4}$; remove from the latest disc a disc of radius $\frac{1}{8}$; add back a disc of radius $\frac{1}{16}$; repeat forever.

HW #5
Find the shaded area.

If $a > 1$, then

$$\sum_{k=0}^{\infty} a^k = \lim_{n\to\infty} \sum_{k=0}^{n} a^k = \lim_{n\to\infty} \frac{1-a^{n+1}}{1-a} = \infty$$

because $a^{+\text{large}} = +\text{large}$ when $a > 1$, and
(Try $(1.2)^{100}$ ...)

$1-a$ is negative, so $\frac{1-a^{n+1}}{1-a} = \frac{1-\text{large}}{-\text{medium}} = +\text{large}$

when n is large.

If $a < -1$, then $\sum_{k=0}^{\infty} a^k = \lim_{n\to\infty} \frac{1-a^{n+1}}{1-a}$

does not exist and is not ∞ or $-\infty$
because for n large, even, $\frac{1-a^{n+1}}{1-a} = +\text{large}$;
for n large, odd, $\frac{1-a^{n+1}}{1-a} = -\text{large}$.

If $a = 1$,

$$\sum_{k=0}^{\infty} a^k = \lim_{n\to\infty} \sum_{k=0}^{n} 1^k = \lim_{n\to\infty} \underbrace{(1^0 + 1^1 + \cdots + 1^n)}_{\substack{n+1 \text{ terms} \\ (k=0 \text{ to } k=n)}}$$

$$= \lim_{n\to\infty} (n+1) = \infty$$

If $a = -1$, $\displaystyle\sum_{k=0}^{\infty} a^k = \lim_{n\to\infty} \frac{1-(-1)^{n+1}}{1-(-1)}$, so

$$\sum_{k=0}^{\infty} a^k = \lim_{n\to\infty} \begin{cases} \dfrac{1-(-1)}{2} & : n \text{ even} \\ \dfrac{1-(+1)}{2} & : n \text{ odd} \end{cases}$$

$$= \lim_{n\to\infty} \begin{cases} 1 & : n \text{ even} \\ 0 & : n \text{ odd} \end{cases}$$

So, $\displaystyle\sum_{k=0}^{\infty} a^k$ does not exist (and is not $\pm\infty$).

Summary of cases:

$$\sum_{k=0}^{\infty} a^k = \begin{cases} \dfrac{1}{1-a} & : -1 < a < 1 \\ \infty & : a \geq 1 \\ \text{undefined} & : a \leq -1 \end{cases}$$