Test 5

#1	#2
50	
50	
42	
38	
36	
35	
31	
30	
30	
30	
28	
28	
28	
27	
25	
25	
median 25	
25	
25	
25	
25	
25	
25	
23	
20	
20	
20	
20	
15	
15	
10	
10	
5	
5	
0	
0	

average = 24.2

Remark on #1 of Test 5:

$$\frac{dx}{dt} + 4x = 3t$$

$$\left(\frac{dx}{dt} + 4x\right)dt = 3t\,dt$$

$$dx + 4x\,dt = 3t\,dt$$

You can't separate the x's & t's this way

Another remark:

$\int 4x$ is nonsense

$\int 4x\,dx = 2x^2 + c$ is easy

$\int 4x\,dt$ makes sense,

but how do you get a formula for it?

$$\int_2^3 4x \approx 4(2 + 2.25 + 2.5 + 2.75) = 38$$

$$\int_2^3 4x\,dx \approx 4(2 + 2.25 + 2.5 + 2.75)(.25)$$

$dx = 0.25$ ✓ $10 \approx 9.5$

$$\int_2^3 4x\,dx = 2x^2\Big|_2^3 = 2(3^2 - 2^2) = \boxed{10}$$

$$dx = 0.01 \Rightarrow \int_2^3 4x \approx 4(2 + 2.01 + 2.02 + \dots + 2.99) \approx 1000$$

$$\int_2^3 4x\,dx \approx 4(2+2.01+\cdots+2.99)(0.01)$$

$$\approx 10 \checkmark$$

$dx = 0.01$

If $\int_2^3 4x$ means anything, it means ∞, but $\int_2^3 4x\,dx = 10$

HW #1 $\quad \frac{dx}{dt}\cos(3t) - x\sin(3t) = \sin 3t$

Find the general solution for x.

#2 $\quad \frac{dx}{dt}\cos(3t) - x\sin(3t) = \sin^3 3t$

Find the solution for x such that $x=10$ when $t=0$.

#3 Assuming 3% inflation, \$1 t years from today is worth $\$(1.03^{-t})$ today. How much is an eternal sequence of monthly payments of \$100, with the first payment today, worth in terms of today's dollars?

What if the payment stopped after the first 600 payments (50 years)?
the first 1200 payments (100 years)?
the first 6000 payments (500 years)?

$0 = ax^2 + bx + c$	(1)	$\cancel{\alpha(t)\ \beta}$ $0 = \alpha(t)\frac{dx}{dt} + \beta(t)x + \gamma(t)$
$0 = x^2 + \frac{b}{a}x + \frac{c}{a}$	(2)	$0 = \frac{dx}{dt} + \frac{\beta(t)}{\alpha(t)}x + \frac{\gamma(t)}{\alpha(t)}$
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	(3)	$\frac{dx}{dt} + \frac{\beta(t)}{\alpha(t)}x = -\frac{\gamma(t)}{\alpha(t)}$
$2p = \frac{b}{a}, \quad q = -\frac{c}{a}$	(4)	$\frac{dp}{dt} = \frac{\beta}{\alpha}, \quad q = -\frac{\gamma}{\alpha}$
$p = b/(2a)$	(5)	$p = \int(\beta/\alpha)\,dt$
$x^2 + 2px = q$	(6)	$\frac{dx}{dt} + \frac{dp}{dt}x = q$
$x^2 + 2px + p^2 = p^2 + q$	(7)	$e^p\left(\frac{dx}{dt} + \frac{dp}{dt}x\right) = e^p q$
$(x+p)^2 = p^2 + q$	(8)	$\frac{d(e^p x)}{dt} = e^p q$
$x + p = \pm\sqrt{p^2+q}$	(9)	$e^p x = \left(\int e^p q\,dt\right) + c$
$x = -p \pm \sqrt{p^2+q}$	(10)	$x = e^{-p}\int e^p q\,dt + ce^{-p}$
$\cancel{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$		$\cancel{x = e^{-\int\frac{\beta}{\alpha}dt}\int e^{\int\frac{\beta dt}{\alpha}}\left(\frac{-\gamma\,dt}{\alpha}\right) + ce^{-\int\frac{\beta}{\alpha}dt}}$
Use any extra info to determine whether $\pm$ is $+$ or $-$.	(11)	Use any extra info to determine what c is.

Convergence: $\sum_{n=0}^{\infty} a_n$ is finite (and exists)

Divergence: $\sum_{n=0}^{\infty} a_n$ does not exist (including the case $= \pm\infty$)

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2} \quad \text{converges}$$

$$\sum_{k=0}^{\infty} 2^k = \lim_{n\to\infty} \sum_{k=0}^{n} 2^k = \underbrace{\lim_{n\to\infty} \frac{1-2^{n+1}}{1-2}} \quad \text{diverges}$$

$$\hookrightarrow \frac{1-\text{big}}{1-2} \to \infty$$

$$\sum_{k=1}^{\infty} k\left(\frac{1}{3}\right)^k = 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{9}\right) + 3\left(\frac{1}{27}\right) + 4\left(\frac{1}{81}\right) + \cdots$$

Does this converge?

$$> \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

M sufficiently large

$$< \underbrace{\cdots}_{\text{finite}} + \frac{1}{(2.9)^M} + \frac{1}{(2.9)^{M+1}} + \frac{1}{(2.9)^{M+2}} + \cdots$$

k	$k\left(\frac{1}{3}\right)^k$		$\left(\frac{1}{2.9}\right)^k$
1	0.3333	$<$	0.3448
2	0.2222	$>$	0.1189
3	0.1111	$>$	0.041
4	0.0493	$>$	0.0141
5	0.0205	$>$	0.0048
6	0.0082	$>$	0.0016
7	0.0032	$>$	0.00058
8	0.0012	$>$	0.0002
20	~~6×10^{-10}~~ 5.7×10^{-9}	$>$	6×10^{-10}

k	$k\left(\frac{1}{3}\right)^k$		$\left(\frac{1}{2.9}\right)^k$
147	1.072×10^{-68}	$>$	1.065×10^{-68}
148	3.600×10^{-69}	$<$	3.674×10^{-69}
149	1.208×10^{-69}	$<$	1.267×10^{-69}
150	4.054×10^{-70}	$<$	4.368×10^{-70}
500	1.376×10^{-233}	$<$	6.324×10^{-232}
5000	1.238×10^{-2382}	$<$	1.023×10^{-2312}

Switch between $k=147$ & $k=148$ to $k\left(\frac{1}{3}\right)^k < \left(\frac{1}{2.9}\right)^k$; never switches again...

Since eventually $0 < k\left(\frac{1}{3}\right)^k < \left(\frac{1}{2.9}\right)^k$,

and $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{2.9}\right)^k = \sum_{k=0}^{\infty}\left(\frac{1}{2.9}\right)^k - \left(\frac{1}{2.9}\right)^0 =$

$$= \frac{1}{1-\frac{1}{2.9}} - 1 = \frac{1}{\frac{1.9}{2.9}} - 1 = \frac{2.9}{1.9} - \frac{1.9}{1.9}$$

$= \frac{1}{1.9}$, we conclude that

$$0 < \underbrace{\sum_{k=1}^{\infty} k\left(\frac{1}{3}\right)^k}_{\text{This actually converges}} < \sum_{k=1}^{\infty} \left(\frac{1}{2.9}\right)^k = \frac{1}{1.9}.$$

finite + error

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error from first 147 terms

To be continued...