Last time, we found numerical evidence that eventually (for $k \geq 148$) $\frac{k}{3^k} < \frac{1}{2.9^k}$

Since $\sum_{k=0}^{\infty} \frac{1}{2.9^k}$ converges,

and $0 < \frac{k}{3^k} < \frac{1}{2.9^k}$ (eventually),

we should expect $\sum_{k=0}^{\infty} \frac{k}{3^k}$

should converge too.

Ratio Test (9.6) ... Motivation:

$$\sum_{k=0}^{\infty} L^k = \frac{1}{1-L} \text{ if } -1<L<1$$

$$\frac{L^{k+1}}{L^k} = L$$

If $\frac{a_{k+1}}{a_k} \approx \pm L$

for k large,

then $a_{k+1} \approx \pm L\, a_k$

$$a_{k+2} = a_k \cdot \frac{a_{k+1}}{a_k} \cdot \frac{a_{k+2}}{a_{k+1}} \approx a_k (\pm L)(\pm L)$$

$$a_{k+3} = a_k \, \frac{a_{k+1}}{a_k} \cdot \frac{a_{k+2}}{a_{k+1}} \cdot \frac{a_{k+3}}{a_{k+2}} \approx a_k (\pm L)(\pm L)(\pm L)$$

It's reasonable to expect that

$\sum_{k=0}^{\infty} a_k$ converges if $\sum_{k=0}^{\infty} L^k$ does.

Here's the ratio test: (9.6)

If $\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|$ exists and

equals some $L < 1$, then

$\sum_{k=0}^{\infty} a_k$ converges.

If $\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = L > 1$ or $= \infty$,

then $\sum_{k=0}^{\infty} a_k$ diverges.

The test is inconclusive in all other cases.

$a_k = \frac{k}{3^k}$ Apply the ratio test.

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{a_{k+1}}{a_k} = \frac{\left((k+1)/3^{k+1}\right)}{(k/3^k)} \cdot \frac{3^k/k}{3^k/k}$$

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)/3^1/k}{1} = \frac{(1+(1/k))/3}{1}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = (1+0)/3 = \frac{1}{3} < 1$$

By the ratio test, $\sum_{k=0}^{\infty} \frac{k}{3^k}$ converges.

Does $\sum_{k=2}^{\infty} \frac{e^k}{2^k \ln k}$ converge?

Ratio test $a_k = \frac{e^k}{2^k \ln k} > 0$, so $|a_k| = a_k$

$$\frac{a_{k+1}}{a_k} = \frac{e^{k+1}/(2^{k+1}\ln(k+1))}{e^k/(2^k \ln k)} \cdot \frac{2^k/e^k}{2^k/e^k}$$

$$\frac{a_{k+1}}{a_k} = \frac{e/(2\ln(k+1))}{1/\ln k} = \frac{e}{2} \cdot \frac{\ln k}{\ln k+1}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{e}{2} \lim_{k\to\infty} \frac{\ln k \nearrow \infty}{\ln k+1 \searrow \infty}$$

$\ln(+\text{large}) = +\text{large}$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{e}{2} \lim_{k\to\infty} \frac{1/k}{1/(k+1)}$$

$$= \frac{e}{2} \lim_{k\to\infty} \frac{k+1}{k} = \frac{e}{2} \lim_{k\to\infty} \left(1+\frac{1}{k}\right)$$

$$= \frac{e}{2}(1+0) = \frac{e}{2} > 1$$

$$\Rightarrow \sum_{k=2}^{\infty} a_k \text{ diverges.}$$

HW which of these converge?

1) $\sum_{k=0}^{\infty} \frac{k+5}{1.7^k}$

2) $\sum_{k=1}^{\infty} (-3)^{2k+1}/k^2$

3) $\sum_{k=17}^{\infty} 4^k/(5^k-k)$

4) $\sum_{k=0}^{\infty} 1/(k!)$

$0! = 1 \quad 1! = 1 \quad 2! = 1\cdot 2 = 2$

$3! = 1\cdot 2\cdot 3 = 6 \quad 4! = 1\cdot 2\cdot 3\cdot 4 = 24$