Last time: ratio test (9.6)
Today: power series (9.7)

As usual, extra practice problems are in the book.

If each $c_0, c_1, c_2, c_3, \ldots$ is a constant, and x is a variable, the an (infinite) power series is

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Example: the geometric series:

$c_k = 1$ for all k

$$\sum_{k=0}^{\infty} (1 \cdot x^k) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

converges to $\frac{1}{1-x}$ if $-1 < x < 1$;

diverges otherwise.

$$\sum_{k=0}^{\infty} (k+1) x^k = (0+1)x^0 + (1+1)x^1 + (2+1)x^2 + (3+1)x^3 + \cdots = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots$$

For which x does $\sum_{k=0}^{\infty} (k+1)x^k$ converge?

For any x, we can apply the ratio test.

$a_k = (k+1)x^k$. The ratio test says that if $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$, then $\sum_{k=0}^{\infty} a_k$ converges; if $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1$, then $\sum_{k=0}^{\infty} a_k$ diverges.

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{(\overbrace{k+2}^{(k+1)+1})|x|^{k+1}}{(k+1)|x|^k} = \frac{k+2}{k+1}|x| = \frac{(k+2)/k}{(k+1)/k}|x|$$

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = |x| \lim_{k\to\infty} \frac{1+2/k}{1+1/k} = |x| \frac{1+0}{1+0}$$

$$= |x|$$

If $\underbrace{|x| < 1}_{-1<x<1}$, then $\sum_{k=0}^{\infty} \underbrace{(k+1)x^k}_{a_k}$ converges;

if $\underbrace{|x| > 1}_{x<-1 \text{ or } x>1}$ then $\sum_{k=0}^{\infty} (k+1)x^k$ diverges;

the ratio test gives no info when $\overbrace{|x| = 1}^{x=\pm 1}$.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

$$\left(\frac{1}{1-x}\right)' = \sum_{k=0}^{\infty} (k+1) x^k = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

$$\left(\frac{1}{1-x}\right)' = \left((1-x)^{-1}\right)' = \frac{d\left((1-x)^{-1}\right)}{dx}$$

$$= \frac{(-1)(1-x)^{-1-1}\, d(1-x)}{dx} = \frac{(-1)(1-x)^{-2}(-dx)}{dx}$$

$$= (1-x)^{-2} = \frac{1}{(1-x)^2}$$

This works assuming the ratio $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$ is < 1 for the x-value you're interested in. In our case, it works for $-1 < x < 1$.

Last time we proved $\sum_{k=0}^{\infty} \frac{k}{3^k}$ converges.

Now we can find what it converges to.

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

$$\sum_{k=0}^{\infty} k/3^k = \frac{0}{3^0} + \frac{1}{3^1} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \cdots$$

$$= \frac{1}{3}\left(0 + 1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \cdots\right)$$

$$= \frac{1}{3}\left(1 + 2\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)^2 + 4\left(\frac{1}{3}\right)^3 + \cdots\right)$$

$$= \frac{1}{3}\sum_{k=0}^{\infty}(k+1)\left(\frac{1}{3}\right)^k = \frac{1}{3}\cdot\frac{1}{\left(1-\frac{1}{3}\right)^2}$$

This works because $-1 < \frac{1}{3} < 1$.

So, $\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \frac{5}{243} + \frac{6}{729} + \cdots$

$$= \frac{1}{3}\cdot\frac{1}{(2/3)^2} = \frac{1}{3}\cdot\frac{1}{4/9} = \frac{1}{4/3} = \frac{3}{4}$$

HW #1 Compute $\displaystyle\sum_{k=0}^{\infty}\frac{(k+2)(k+3)}{4^k}$ exactly.

More generally, we can series of powers of a function:

$$\sum_{k=0}^{\infty} \frac{\sin^k x}{2^k} = \sum_{k=0}^{\infty} \left(\frac{1}{2^k}\right) \cancel{g}(\sin(x))^k$$

General form: $\displaystyle\sum_{k=0}^{\infty} c_k (g(x))^k$

Use ratio test: $a_k = (\sin^k x)/2^k$

$$\left|\frac{(\sin^{k+1} x)/2^{k+1}}{(\sin^k x)/2^k}\right| = \left|\frac{a_{k+1}}{a_k}\right|$$

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{|\sin x|/2}{1} \qquad \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \frac{|\sin x|}{2}$$

$-1 \le \sin x \le 1$, so $|\sin x| \le 1$,

so $\dfrac{|\sin x|}{2} \le \dfrac{1}{2} < 1$, so, by the ratio test, $\displaystyle\sum_{k=0}^{\infty} \frac{\sin^k x}{2^k}$ converges

~~and~~ for all x.

$$\sum_{k=0}^{\infty} \frac{\sin^k x}{2^k} = \sum_{k=0}^{\infty} \left(\frac{\sin x}{2}\right)^k = \frac{1}{1-\frac{\sin x}{2}}$$

$\ln 3 = ?$ You can approximate

$$\ln 3 = \int_1^3 \frac{dx}{x} \text{ with}$$

Simpson's Rule, etc...

You can also approximate $\ln 3$ with series.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{for } -1<x<1$$

$$\int \frac{-du}{u} = \int \frac{dx}{1-x} = \sum_{k=0}^{\infty} \int x^k \, dx \quad \text{for } \_\_\_\_\_$$

$u = 1-x$

$du = -dx$

$-du = dx$

$$-\ln|u| = -\ln|1-x|$$

If we plug in $x = \frac{2}{3}$, then

$$-\ln|1-x| = -\ln\left|1-\frac{2}{3}\right| = -\ln\frac{1}{3} = \ln 3$$

If $\sum_{k=0}^{\infty} \int x^k \, dx$ converges, at $x = \frac{2}{3}$, to $\ln 3$, then we can approximate $\ln 3$ by finite initial subsums like $\sum_{k=0}^{10} \int x^k \, dx$, $\sum_{k=0}^{100} \int x^k \, dx$, etc.

What about the "+c" from the integrals? We need to pick limits of integration to avoid this problem.

$$\int_0^{2/3} \frac{dx}{1-x} = \int_1^{1/3} \frac{-du}{u} = -\ln\frac{1}{3} - (\underbrace{-\ln 1}_{0})$$

$$\int_0^{2/3} \frac{dx}{1-x} = \ln 3$$

HW #2 Check that the ratio test result for $\sum_{k=0}^{\infty} \int_0^x t^k \, dt$ is "convergence" for all x with $-1 < x < 1$.

~~Find a series app~~

Then write out the first five terms of $\sum_{k=0}^{\infty} \int_0^x t^k \, dt$ when $x = \frac{2}{3}$. This is an estimate of $\ln 3$.

HW #3 a) Use $\arctan x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \frac{dt}{1-(-t^2)}$ to express $\frac{\pi}{6} = \arctan \frac{1}{\sqrt{3}}$ as an infinite series.

b) Write out the first five terms.

c) Check that your series converges using the ratio test.