$$f(x) = \sum_{k=0}^{\infty} c_k (g(x))^k$$ very generalized power series

$c_0, c_1, c_2, \ldots$ constants

Abbreviation: x is "ratio-safe"

if $\displaystyle\lim_{k\to\infty} \left| \frac{c_{k+1} (g(x))^{k+1}}{c_k (g(x))^k} \right| < 1$

or $g(x) = 0$.

<u>Fact 1.</u> If g is continuous on $[a,b]$ and all x in $[a,b]$ are ratio-safe, then $\displaystyle\int_a^b f(x)\,dx = \sum_{k=0}^{\infty} c_k \int_a^b g(x)^k\,dx$.

Fact 2. If g' is continuous on an open interval (a,b), and all x in (a,b) are ratio-safe, then, for all x in (a,b), $\displaystyle\frac{df}{dx} = \sum_{k=0}^{\infty} c_k \frac{d(g^k)}{dx}$

and x is ratio-safe for the new series

$$\frac{1}{1\cdot 5} - \frac{1}{2\cdot 5^2} + \frac{1}{3\cdot 5^3} - \frac{1}{4\cdot 5^4} + \cdots = ?$$

$k=0 \qquad k=1 \qquad k=2 \qquad k=3$

$$= \frac{(-1)^0}{(0+1)5^{0+1}} + \frac{(-1)^1}{(1+1)5^{1+1}} + \frac{(-1)^2}{(2+1)5^{2+1}} + \frac{(-1)^3}{(3+1)5^{3+1}} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)5^{k+1}} = -\sum_{k=0}^{\infty} \frac{1}{k+1}\left(\frac{-1}{5}\right)^{k+1}$$

Alternative:

$k=1 \qquad k=2 \qquad k=3 \qquad k=4$

$$= \frac{-(-1)^1}{1\cdot 5^1} + \frac{-(-1)^2}{2\cdot 5^2} + \frac{-(-1)^3}{3\cdot 5^3} + \frac{-(-1)^4}{4\cdot 5^4} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{-(-1)^k}{k\cdot 5^k} = \sum_{k=1}^{\infty} \frac{-1}{k}\left(\frac{-1}{5}\right)^k$$

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots = \sum_{k=0}^{\infty} t^k$$

for $-1 < t < 1$

$$\int_0^x \frac{dt}{1-t} = \sum_{k=0}^{\infty} \int_0^x t^k\, dt \qquad \text{for } -1 < x < 1$$

$u = 1 - t \quad du = -dt \qquad -du = dt$

$t = 0 \Rightarrow u = 1 - 0 = 1 \qquad t = x \Rightarrow u = 1 - x$

$$\int_1^{1-x} \frac{-du}{u} = \sum_{k=0}^{\infty} \left(\frac{t^{k+1}}{k+1} \Big|_0^x \right) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

$x = -\frac{1}{5}$ can be plugged in $\left(-1 < -\frac{1}{5} < 1\right)$

$$\int_1^{1-(-1/5)} \frac{-du}{u} = \sum_{k=0}^{\infty} \frac{(-1/5)^{k+1}}{k+1}$$

$$-\int_1^{6/5} \frac{du}{u} = \frac{1}{1 \cdot 5} - \frac{1}{2 \cdot 5^2} + \frac{1}{3 \cdot 5^3} - \cdots$$

$$= -\ln|u| \Big|_1^{6/5} = -\ln\frac{6}{5} - (\underbrace{-\ln 1}_{0}) = -\ln\frac{6}{5}$$

$$-\ln\frac{6}{5} = -\frac{1}{5} + \frac{1}{50} - \frac{1}{375} + \frac{1}{4 \cdot 625} - \cdots$$

~~0.18232~~

$$\underbrace{\ln\frac{6}{5}}_{0.182322} = \underbrace{\frac{1}{5} - \frac{1}{50} + \frac{1}{375} - \frac{1}{4 \cdot 625} + \frac{1}{5 \cdot 3125}}_{0.182331}$$

$$- \frac{1}{6 \cdot 15625} + \frac{1}{7 \cdot 5^7} - \frac{1}{8 \cdot 5^8} + \cdots$$

HW #1

$$\frac{4}{1\cdot 2\cdot 1} - \frac{16}{2\cdot 3\cdot 5} + \frac{64}{3\cdot 4\cdot 25} - \frac{256}{4\cdot 5\cdot 125}$$

$$+ \frac{1024}{5\cdot 6\cdot 625} - \frac{4^6}{6\cdot 7\cdot 5^5} + \frac{4^7}{7\cdot 8\cdot 5^6} - \frac{4^8}{8\cdot 9\cdot 5^7}$$

$$+ \frac{4^9}{9\cdot 10\cdot 5^8} - \frac{4^{10}}{10\cdot 11\cdot 5^9} + \cdots = ?$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$ Ratio test: $a_k = x^k/k!$

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{|x|^{k+1}/(k+1)!}{|x|^k/k!} = \frac{|x|}{(k+1)!/k!} = \frac{|x|}{k+1}$$

$$\frac{5!}{4!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{1\cdot 2\cdot 3\cdot 4} = 5$$

$$\frac{(k+1)!}{k!} = \frac{1\cdot 2\cdot 3\cdot \cdots \cdot k\cdot k+1}{1\cdot 2\cdot 3\cdot \cdots \cdot k} = k+1$$

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{|x|}{k+1} = 0 < 1 \Rightarrow$$ converge for all x

$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for all x.

(& the ratio test result is < 1) for all $x \neq 0$.

$$f'(x) = \sum_{k=0}^{\infty} \left(\frac{x^k}{k!}\right)' = \sum_{k=0}^{\infty} \frac{k x^{k-1}}{k!}$$

$$f'(x) = \frac{0x^{-1}}{0!} + \frac{1x^0}{1!} + \frac{2x^1}{2!} + \frac{3x^2}{3!} + \cdots$$

$(0! = 1)$ $\quad f'(x) = x^0 + x^1 + \frac{x^2}{2!} + \cdots$

The next term is $\frac{4x^3}{4!} = \frac{4x^3}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{x^3}{3!}$

$x^0 = \frac{x^0}{0!}$ $\quad x^1 = \frac{x^1}{1!}$, so

$$f'(x) = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$f'(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = f(x)$$

The ~~th~~ only solutions to $f' = f$ are $f(x) = Ce^x$.

Here's why:

$$f' = \frac{df}{dx} = f \Rightarrow \int \frac{df}{f} = \int dx \Rightarrow \ln|f| = x + c_0$$

$$\Rightarrow |f| = e^{x+c_0} = e^x e^{c_0} \Rightarrow f = \underbrace{\pm e^{c_0}}_{C} e^x$$

Therefore, $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} = Ce^x$

~~[illegible]~~ for some C.

Plug in $x = 0$:

$$Ce^0 = C; \quad \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

$$= 1 + 0 + 0 + 0 + 0 + \cdots$$

when $x = 0$

So, $C = 1$.

So, $\displaystyle e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$e = e^1 = \sum_{k=0}^{\infty} \frac{1^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!}$$

HW #2 Express $\int_0^1 e^{-x^2}\,dx$

as an infinite series.