Today: alternating series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \leftarrow \text{when } -1<x<1$$

when $-1<t<1$

$$\sum_{k=0}^{\infty} \int_0^t x^k\,dx = \int_0^t \frac{dx}{1-x} = -\ln(1-t) \Rightarrow \ln\left(\frac{1}{1-t}\right)$$

At $t=-1$, we have:

$$\sum_{k=0}^{\infty} \underbrace{\int_0^{-1} x^k\,dx}_{\left.\frac{x^{k+1}}{k+1}\right|_0^{-1}} \overset{?}{=} -\ln(1-(-1)) = -\ln 2$$

$$\underbrace{\left.\frac{x^{k+1}}{k+1}\right|_0^{-1}} = \frac{(-1)^{k+1}}{k+1} - \frac{0^{k+1}}{k+1}$$

$$\frac{(-1)^1}{1} + \frac{(-1)^2}{2} + \frac{(-1)^3}{3} + \frac{(-1)^4}{4} + \cdots \overset{?}{=} -\ln 2$$

$k=0$ $\quad k=1$ $\quad k=2$ $\quad k=3$

$$\underbrace{\frac{-1}{1} + \frac{1}{2} + \frac{-1}{3} + \frac{1}{4} + \cdots}_{\text{does this even converge?}} \overset{?}{=} -\ln 2$$

Remember that $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges by the integral test.

Try ratio test: $a_k = \frac{(-1)^{k+1}}{k+1}$

$$\frac{a_{k+1}}{a_k} = \frac{(-1)^{k+2}/(k+2)}{(-1)^{k+1}/(k+1)} = \frac{(-1)/(k+2)}{1/(k+1)}$$

$$\frac{a_{k+1}}{a_k} = \frac{-1}{(k+2)/(k+1)} = \frac{-1}{((k+1)+1)/(k+1)}$$

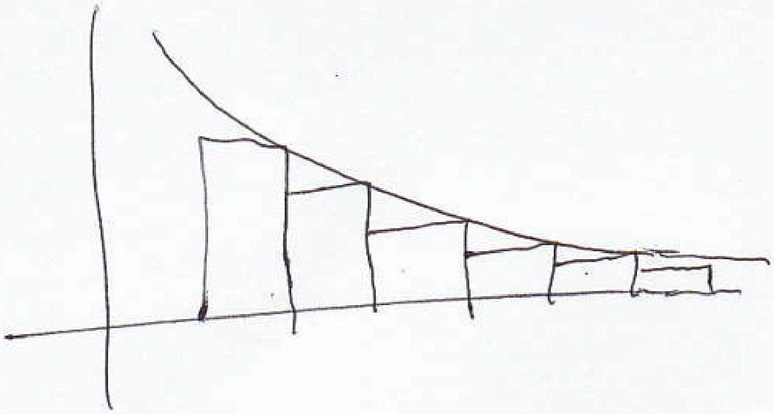
$$\frac{a_{k+1}}{a_k} = \frac{-1}{1+1/(k+1)} \to \frac{-1}{1+0}$$

as $k \to \infty$

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{-1}{1+0}\right| = 1$$

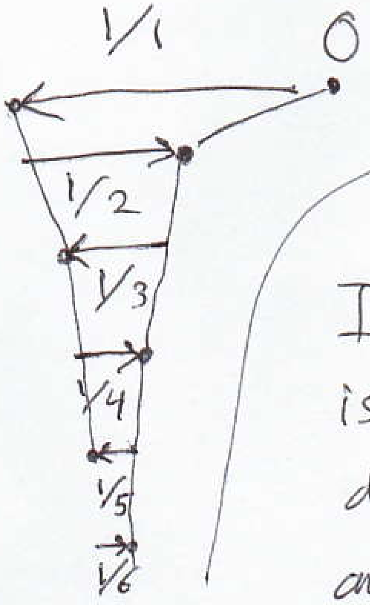
Inconclusive.

We can apply the integral test when we have $\sum_{n=0}^{\infty} f(n)$ where f is positive and decreasing ($\searrow$).



This doesn't apply to

$$\frac{-1}{1} + \frac{1}{2} + \frac{-1}{3} + \frac{1}{4} + \frac{-1}{5} + \frac{1}{6} + \cdots$$



This does converge.

Alternating Series Test

If $b_0, b_1, b_2, b_3, \ldots$ is a positive and decreasing ($\searrow$) sequence, and $\lim_{k\to\infty} b_k = 0$,

then $\underbrace{\sum_{k=0}^{\infty} (-1)^k b_k}_{b_0 - b_1 + b_2 - b_3 + b_4 - \cdots}$ converges.

$$-\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots$$

$$= (-1)\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right)$$

$$= (-1)\sum_{k=0}^{\infty} (-1)^k/(k+1) \qquad b_k = \frac{1}{k+1}$$

So, yes, $-\frac{1}{1}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots$

converges to $-\ln 2$.

So, $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$

converges to $\ln 2$.

For a convergent alternating series, there are easy estimates of what it converges to.

$\xrightarrow{1}$ $\qquad \ln 2 < 1$

$\xleftarrow{1/2}$ $\qquad 1-\frac{1}{2} < \ln 2$

$\xrightarrow{1/3}$ $\qquad \ln 2 < 1-\frac{1}{2}+\frac{1}{3}$

$\xleftarrow{1/4}$ $\qquad 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4} < \ln 2$

$\vdots$

$\ln 2$ $\qquad \ln 2 < 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$

$$\overset{k=0}{\frac{1}{2}} - \overset{k=1}{\frac{2}{3}} + \overset{k=2}{\frac{3}{4}} - \frac{4}{5} + \frac{5}{6} - \frac{6}{7} + \cdots$$

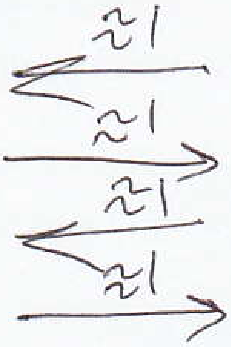
$$\cdots + \frac{(-1)^k(k+1)}{k+2} + \cdots$$

For k large, $\frac{k+1}{k+2} = \frac{(k+1)/k}{(k+2)/k}$

$$= \frac{1+1/k}{1+2/k} \approx \frac{1+0}{1+0}, \text{ so } \frac{(-1)^k (k+1)}{(k+2)} \approx \pm 1,$$

so you'll oscillate forever with

step size ≈ 1.



$$\lim_{k\to\infty} \frac{k+1}{k+2} = 1 \neq 0.$$

Let's estimate e^{-1}.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

$$e^{-1} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$

$$= \frac{1}{1} - \frac{1}{1} + \frac{1}{1\cdot 2} - \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4}$$

$$- \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots$$

$$1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120} < e^{-1} < 1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}$$

$$\cdots < e^{-1} < 1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720} \approx .368$$

$$\frac{11}{30} \approx .3666\ldots < e^{-1} < .375$$

HW #1

Use alternating series to get upper and lower bounds for $\arctan(1) = \frac{\pi}{4}$.

Hint: $\arctan(1) = \int_0^1 \frac{dx}{1-(-x^2)}$

Get bounds less than 0.05 apart.

HW #2 Does $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ converge?

HW #3 Does $\sum_{k=1}^{\infty} \frac{(-1)^k k}{1+k^2}$ converge?

HW #4 Does $\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{1+k^2}$ converge?