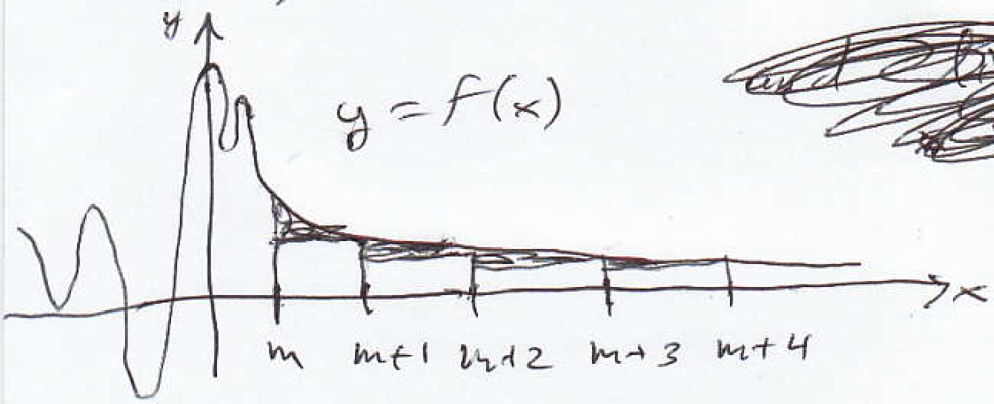
Past: estimates of alternating ~~series~~ series

Today: estimates of series using integrals

Future: estimates of series using geometric series
(Wednesday)

Situation: $f(x)$ is positive, at least eventually



$$\sum_{k=m+1}^{\infty} f(k) < \int_m^{\infty} f(x)\,dx$$

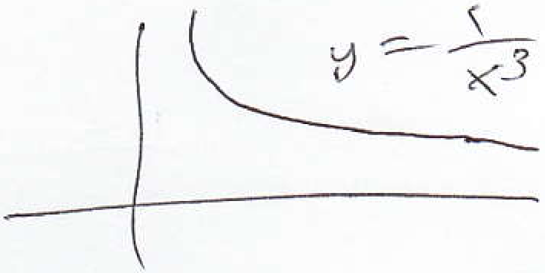
Since f is eventually positive, there are not oscillations going on forever, so if $\int_m^{\infty} f(x)\,dx$ converges, then so does $\sum_{m+1}^{\infty} f(k)$.

Idea: estimate $\sum_{k=1}^{\infty} f(k)$ with $\sum_{k=1}^{m} f(k)$

$$\underbrace{\sum_{k=1}^{\infty} f(k)}_{\substack{\text{exact}\\\text{value}}} = \underbrace{\sum_{k=1}^{m} f(k)}_{\substack{\text{estimate}\\\text{(underestimate)}}} + \underbrace{\sum_{k=m+1}^{\infty} f(k)}_{\substack{\text{remainder}\\\text{(positive)}}}$$

(underestimate) $\Longleftarrow$ (positive)

$$\underbrace{\sum_{k=1}^{m} f(k)}_{\substack{\text{lower}\\\text{bound}}} < \underbrace{\sum_{k=1}^{\infty} f(k)}_{\substack{\text{true}\\\text{value}}} < \underbrace{\sum_{k=1}^{m} f(k) + \int_m^{\infty} f(x)\,dx}_{\text{upper bound}}$$



$$f(x) = \frac{1}{x^3}$$

$\uparrow$

positive when $x > 0$

$$f'(x) = (x^{-3})' = -3x^{-4} < 0$$

so f is $\searrow$.

$$\int_m^{\infty} f(x)\,dx = \int_m^{\infty} x^{-3}\,dx$$

$$= \lim_{t\to\infty} \int_m^t x^{-3}\,dx = \lim_{t\to\infty} \left.\frac{x^{-2}}{-2}\right|_m^t$$

$$= \lim_{t\to\infty} \frac{-1}{2x^2}\Big|_m^t = \lim_{t\to\infty}\left(\frac{-1}{2t^2} - \frac{-1}{2m^2}\right)$$

$$= 0 - \frac{-1}{2m^2} = \frac{1}{2m^2} \Rightarrow \int_m^\infty \frac{1}{x^3}\,dx$$

converges $\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^3}$ converges too.

m	lower bound	upper bound
1	$\frac{1}{1^3}$	$\frac{1}{1^3} + \frac{1}{2\cdot 1^2}$
2	$1/1^3 + 1/2^3$	$1/1^3 + 1/2^3 + 1/(2\cdot 2^2)$
3	$1/1^3 + 1/2^3 + 1/3^3$	$1/1^3 + 1/2^3 + 1/3^3 + 1/(2\cdot 3^2)$
4	$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3}$	$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{2\cdot 4^2}$
m	$\sum_{k=1}^{m} 1/k^3$	$\sum_{k=1}^{m} \frac{1}{k^3} + \underbrace{\int_m^\infty \frac{dx}{x^3}}_{1/(2m^2)}$

HW: Estimate following sums with lower & upper bounds < 0.0005 apart:

$$\sum_{k=0}^{\infty} \frac{k}{(1+k^2)^3}, \quad \sum_{k=1}^{\infty} \frac{3}{k^4}, \quad \sum_{k=0}^{\infty} \frac{5}{(k+5)^5}$$

$$\sum_{k=1}^{\infty} \frac{\ln k}{k} \qquad \int_m^{\infty} \frac{\ln x}{x}\, dx$$

$\frac{\ln x}{x}$ is eventually $+$, $\searrow$

HW # 2: Check that

$$\int_m^{\infty} \frac{\ln x}{x}\, dx = \infty, \cancel{\text{that}} \text{ for any } m > 0.$$

Conclude that

$\sum_{k=1}^{\infty} \frac{\ln k}{k}$ diverges.

$$x > 1 \Rightarrow \ln x > 0, \text{ so } \frac{\ln x}{x} > 0$$

$$\left(\frac{\ln x}{x}\right)' = \frac{(\ln' x)\, x - (\ln x)\, x'}{x^2} = \frac{\left(\frac{1}{x}\right) x - (\ln x) 1}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

$$x > e \Rightarrow \ln x > \ln e = 1$$

$$\frac{\ln x}{x} \searrow \Leftarrow \frac{1 - \ln x}{x^2} < 0 \Leftarrow 1 - \ln x < 0$$