Today: ① another "geometric" estimate

② Taylor series (9.10, 9.11)

$$\left(\frac{1}{3}\right)^1 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \left(\frac{4}{9}\right)^4 + \left(\frac{5}{11}\right)^5 + \cdots = ?$$

$$= \underbrace{\left(\frac{1}{2\cdot 1+1}\right)^1}_{k=1} + \underbrace{\left(\frac{2}{2\cdot 2+1}\right)^2}_{k=2} + \underbrace{\left(\frac{3}{2\cdot 3+1}\right)^3}_{k=3} + \cdots$$

$$= \underbrace{\sum_{k=1}^{\infty} \left(\frac{k}{2k+1}\right)^k}_{\text{converges}}$$

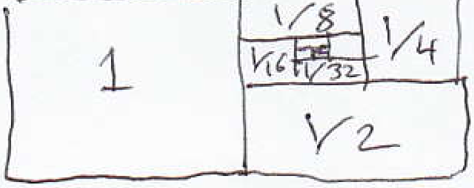
HW: Check that the ratio test result is 1/2.

$$\frac{k}{2k+1} < \frac{k}{2k} = \frac{1}{2}$$

$$\underbrace{\sum_{k=1}^{m} \left(\frac{k}{2k+1}\right)^k}_{\text{lower bound}} < \sum_{k=1}^{\infty} \left(\frac{k}{2k+1}\right)^k = \sum_{k=1}^{m} \left(\frac{k}{2k+1}\right)^k + \underbrace{\sum_{k=m+1}^{\infty} \left(\frac{k}{2k+1}\right)^k}_{\substack{\text{remainder} \\ \text{(positive)}}}$$

$$\sum_{k=m+1}^{\infty} \left(\frac{k}{2k+1}\right)^k < \sum_{k=m+1}^{\infty} \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{m+1} + \left(\frac{1}{2}\right)^{m+2} + \left(\frac{1}{2}\right)^{m+3} + \cdots$$

$$= \left(\frac{1}{2}\right)^{m+1} \underbrace{\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right]}_{1/(1-1/2) \,=\, 1/(1/2) \,=\, 2} = \left(\frac{1}{2}\right)^m = \frac{1}{2^m}$$



$= 2$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ if } |x|<1$$

$$\underbrace{\sum_{k=1}^{m} \left(\frac{k}{2k+1}\right)^k}_{\text{lower bound}} < \sum_{k=1}^{\infty} \left(\frac{k}{2k+1}\right)^k < \underbrace{\sum_{k=1}^{m} \left(\frac{k}{2k+1}\right)^k + \frac{1}{2^m}}_{\text{upper bound}}$$

If, say, $m = 10$, then the spread is $\frac{1}{2^{10}} = \frac{1}{1024} < \frac{1}{1000} = 0.001$.

HW #2 Estimate $\sum_{k=1}^{\infty} \left(\frac{k}{3k-1}\right)^k$ with a spread < 0.002.

Taylor series intro:

$$\begin{aligned}
p(x) &= A + Bx + Cx^2 + Dx^3 + Ex^4 \\
p'(x) &= \quad B + 2Cx + 3Dx^2 + 4Ex^3 \\
p''(x) &= \quad 2C + 6Dx + 12Ex^2 \\
p'''(x) &= \quad 6D + 24Ex \\
p''''(x) &= \quad 24E
\end{aligned}$$

$$p(0) = A + B\cdot 0 + C\cdot 0^2 + D\cdot 0^3 + E\cdot 0^4 = A$$

$$p'(0) = B = (1!)B \qquad p(0) = \underbrace{(0!)}_{1} A$$

$$p''(0) = 2C = (2!)C$$

$$p'''(0) = 6D = (3!)D$$

$$p''''(0) = 24E = (4!)E$$

$$A = \frac{p^{(0)}(0)}{0!} \qquad B = \frac{p^{(1)}(0)}{1!} \qquad C = \frac{p^{(2)}(0)}{2!}$$

$$D = \frac{p^{(3)}(0)}{3!} \qquad E = \frac{p^{(4)}(0)}{4!}$$

$$p(x) = A + Bx + Cx^2 + Dx^3 + Ex^4$$

$$p(x) = \frac{p^{(0)}(0)}{0!}x^0 + \frac{p^{(1)}(0)}{1!}x^1 + \frac{p^{(2)}(0)}{2!}x^2 + \frac{p^{(3)}(0)}{3!}x^3 + \frac{p^{(4)}(0)}{4!}x^4$$

$$p(x) = \sum_{k=0}^{4} \frac{p^{(k)}(0)}{k!}x^k$$

$$\sin(x) = \underbrace{\sum_{k=0}^{4}\left(\frac{\sin^{(k)}(0)}{k!}\right)x^k}_{\text{Taylor series at } x=0 \text{ up through 4th degree}} + \underbrace{\int_0^x (x-t)^4 \frac{\sin^{(5)} t}{4!}\,dt}_{\text{remainder } R_4}$$

remainder is small when x is small.

$$\underbrace{\frac{\sin^{(0)}(0)}{0!}x^0}_{\frac{\sin(0)}{1}\cdot 1} + \underbrace{\frac{\sin^{(1)}(0)}{1!}x^1}_{\frac{\cos(0)}{1}x} + \underbrace{\frac{\sin^{(2)}(0)}{2!}x^2}_{\frac{-\sin(0)}{2}x^2}$$

$$+ \underbrace{\frac{\sin^{(3)}(0)}{3!}x^3}_{\frac{-\cos(0)}{6}x^3} + \underbrace{\frac{\sin^{(4)}(0)}{4!}x^4}_{\frac{\sin(0)}{24}x^4}$$

$$= x - \frac{x^3}{6}$$

For small x, $\sin(x) \approx x - \frac{x^3}{6}$.

(More on Taylor series next week.)