ABEL'S THEOREM FOR SERIES OF POWERS OF FUNCTIONS

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Theorem 1. *Let $g(x)$ be a real function continuous on a nonempty open interval (a,b) (and possibly continuous elsewhere); let $c_0, c_1, c_2, \ldots$ be a sequence of constants. Suppose that $\lim_{k\to\infty}\left|\frac{c_{k+1}g(x)^{k+1}}{c_k g(x)^k}\right|$ exists and is < 1 for all x in (a,b), except where $g(x) = 0$.*

If $f(x)$ and $g(x)$ are continuous from the left at b and $f(x) = \sum_{k=0}^{\infty} c_k(g(x))^k$ for all x in (a,b), then $f(b) = \sum_{k=0}^{\infty} c_k(g(b))^k$.

Likewise, if $f(x)$ and $g(x)$ are continuous from the right at a and $f(x) = \sum_{k=0}^{\infty} c_k(g(x))^k$ for all x in (a,b), then $f(a) = \sum_{k=0}^{\infty} c_k(g(a))^k$.

Proof. By symmetry, we only need to prove the first half of the theorem; the second half follows the first half applied to $p(x) = g(a+b-x)$ in place of $g(x)$.

Let $S(x) = \sum_{k=0}^{\infty} c_k(g(x))^k$. If $g(x) = 0$ on all of (a,b), then $f(x) = c_0$ on all of (a,b), and, by continuity, $g(b) = 0$ and $f(b) = c_0$, so $f(b) = c_0 = S(b)$. Therefore, we may assume $g(x) \neq 0$ for some x in (a,b). Hence, at such an x, $1 > \lim_{k\to\infty}\left|\frac{c_{k+1}g(x)^{k+1}}{c_k g(x)^k}\right| = |g(x)| \lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|$; let $L = \lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|$.

Choose q in (a,b). If $L = 0$, then the ratio-test limit for $S(x)$ will be 0, or $g(x) = 0$, for all x in $[q,b]$, so $S(x)$ will be continuous on $[q,b]$, so $S(b) = \lim_{x\to b^-} S(x) = \lim_{x\to b^-} f(x) = f(b)$. Therefore, we may assume $L > 0$. Similarly, if $|g(b)| < 1/L$, then the ratio-test limit for $S(x)$ will be < 1, or $g(x) = 0$, for all x in $[q,b]$, so again $S(b) = f(b)$. Therefore, we may assume $g(b) \geq 1/L$.

If $|g(b)| > 1/L$, then, by the Intermediate Value Theorem, if we choose r in $(1/L, |g(b)|)$, then $|g(y)| = r$ for some y in (q,b), for $|g|$ is continuous on $[q,b]$ and $|g(q)| < 1/L < r < |g(b)|$. However, in this case $g(y) \neq 0$ and $L|g(y)| > 1$, in contradiction with the ratio-test limit for $S(x)$ being < 1 for all x in (a,b) where $g(x) \neq 0$. Therefore, $|g(b)| = 1/L$.

With the easier cases dispensed with, our strategy will now be to prove that $S(b)$ is infinitely close to $S(x)$ for some x in (a,b) that is infinitely close to b. Since $S(x) = f(x)$ and f is continuous from the left at b, $x \approx b$ will imply that $S(b) \approx f(x) \approx f(b)$. Since $S(b)$ and $f(b)$ are both reals, they can only be infinitely close if they are equal, which is exactly what we wish to prove. Therefore, we just need to find x such that $b \approx x < b$ and $S(x) \approx S(b)$.

Set $d_k = c_k(g(b))^k$ and $h(x) = g(x)/g(b)$, so that $S(x) = \sum_{k=0}^{\infty} d_k(h(x))^k$, $h(b) = 1$, and $S(b) = \sum_{k=0}^{\infty} d_k$. Since $h(b) = 1$ and $h(x) = g(x)/g(b)$ is continuous on $(a,b]$, if J is an infinite positive hyperinteger, then $h(b - 1/J) \approx h(b) = 1$, so $h(b-1/J) > 0$. Fixing such a J, let I be the greatest hyperinteger less than J for which there is $h(b-z) \leq 0$ for some positive $z < 1/I$; if no such hyperinteger exists, set $I = 1$. Since $h(b-z) \approx h(b) = 1$ for all positive $z \approx 0$, I cannot be infinite, so I is finite and, setting $t = b - 1/I$, $h(x) > 0$ for all x in $(t,b]$. We will use this fact later.

Set $s_k = d_0 + d_1 + \cdots + d_{k-1} + d_k$, so that $s_k - s_{k-1} = d_k$ and $S(b) = \lim_{n\to\infty} s_n$. Therefore,

$$\begin{aligned}
\sum_{k=0}^{n} d_k h^k &= d_0 + d_1 h + d_2 h^2 + d_3 h^3 + \cdots + d_{n-1}h^{n-1} + d_n h^n \\
&= s_0 + (s_1 - s_0)h + (s_2 - s_1)h^2 + (s_3 - s_2)h^3 + \cdots + (s_{n-1} - s_{n-2})h^{n-1} + (s_n - s_{n-1})h^n \\
&= s_0(1-h) + s_1(h-h^2) + s_2(h^2 - h^3) + \cdots + s_{n-1}(h^{n-1} - h^n) + s_n h^n \\
&= (1-h)(s_0 + s_1 h + s_2 h^2 + \cdots + s_{n-1}h^{n-1}) + s_n h^n \\
&= (1-h)\sum_{k=0}^{n-1} s_k h^k + s_n h^n
\end{aligned}$$

For all x in (a,b), $|h(x)| = \left|\frac{g(x)}{g(b)}\right| = |g(x)L| < 1$, so $\lim_{n\to\infty} h^n(x) = 0$, so

$$\begin{aligned}
S(x) = \lim_{n\to\infty}\sum_{k=0}^{n} d_k h^k(x) &= \lim_{n\to\infty}\left((1-h(x))\sum_{k=0}^{n-1} s_k h^k(x) + s_n h^n(x)\right) \\
&= (1-h(x))\sum_{k=0}^{\infty} s_k h^k(x) + S(b)\cdot 0
\end{aligned}$$

Therefore, again using the fact that $|h(x)| < 1$,

$$\begin{aligned}
S(x) - S(b) &= (1-h(x))\sum_{k=0}^{\infty} s_k h^k(x) - S(b) \\
&= (1-h(x))\left(\sum_{k=0}^{\infty} s_k h^k(x) - \frac{S(b)}{1-h(x)}\right) \\
&= (1-h(x))\left(\sum_{k=0}^{\infty} s_k h^k(x) - S(b)\sum_{k=0}^{\infty} h^k(x)\right) \\
&= (1-h(x))\sum_{k=0}^{\infty}(s_k - S(b))h^k(x) \\
&= (1-h(x))\sum_{k=0}^{M-1}(s_k - S(b))h^k(x) + (1-h(x))\sum_{k=M}^{\infty}(s_k - S(b))h^k(x)
\end{aligned}$$

for all positive M.

Choose M to be infinitely large. Let N be the greatest hyperinteger (if it exists) for which $|s_k - S(b)| < \frac{1}{N}$ for all $k \geq M$. Since $\lim\limits_{n\to\infty} s_n = S(b)$, $s_k \approx S(b)$ for all infinite k, so $s_k \approx S(b)$ for all $k \geq M$, so N exists and must be infinite. Therefore, for all x in (t, b) (where $0 < h(x) < 1$),

$$\begin{aligned}
\left|(1-h(x)) \sum_{k=M}^{\infty} (s_k - S(b)) h^k(x)\right| &< \left|\frac{1-h(x)}{N}\right| \sum_{k=M}^{\infty} |h^k(x)| \\
&= \frac{1-h(x)}{N}(h^M(x) + h^{M+1}(x) + h^{M+2}(x) + \cdots) \\
&= \frac{(1-h(x))h^M(x)}{N}(1 + h(x) + h^2(x) + \cdots) \\
&= \frac{(1-h(x))h^M(x)}{N}\frac{1}{1-h(x)} \\
&= \frac{h^M(x)}{N}.
\end{aligned}$$

Since $|h(x)| < 1$, $|h^M(x)| < 1$, so $\left|(1-h(x)) \sum_{k=M}^{\infty} (s_k - S(b)) h^k(x)\right| < \frac{1}{N}$. Therefore,

$$(1-h(x)) \sum_{k=M}^{\infty} (s_k - S(b)) h^k(x) \approx 0.$$

Hence, $S(x) - S(b) \approx (1-h(x)) \sum_{k=0}^{M-1} (s_k - S(b)) h^k(x) + 0$.

Let P be the maximum value appearing in the (hyper)list $|s_0 - S(b)|, |s_1 - S(b)|, \ldots, |s_{M-1} - S(b)|$. $|s_i - S(b)|$ is finite for all finite i, and $s_i \approx S(b)$ for all infinite i, so $|s_i - S(b)|$ is finite for all i. Therefore, P is finite, so we can use P to establish a useful upper bound on $|S(x) - S(b)|$:

$$\begin{aligned}
|S(x) - S(b)| &\approx \left|(1-h(x)) \sum_{k=0}^{M-1} (s_k - S(b)) h^k(x)\right| \\
&\leq (1-h(x)) P \sum_{k=0}^{M-1} h^k(x) \\
&= (1-h(x)) P \frac{1-h^M(x)}{1-h(x)} \\
&= P(1-h^M(x)).
\end{aligned}$$

Our goal is to show that $S(x) \approx S(b)$ for some x with $b \approx x < b$, so if we can show that $h^M(x) \approx 1$ for some such x, then we will have $|S(x) - S(b)| \approx P(1-h^M(x)) \approx 0$, which will imply $S(x) \approx S(b)$. When $b \approx x < b$, $h(x) \approx h(b) = 1$, so $h^m(x) \approx 1$ for all real m. Therefore, for every positive real δ and real m, there is a hyperreal x such that $b - \delta < x < b$ and $|1 - h^m(x)| < \delta$. By Transfer, for every positve real δ and real m, there is a real x such that $b - \delta < x < b$ and $|1 - h^m(x)| < \delta$. By Transfer again, for every positive hyperreal δ and hyperreal m, there is a hyperreal x such that $b - \delta < x < b$ and $|1 - h^m(x)| < \delta$. In particular, if we choose δ to be positive infinitesimal, then there is a hyperreal x such that $b - \delta < x < b$ and $|1 - h^M(x)| < \delta$. Thus, there is a hyperreal x such that $b \approx x < b$ and $h^M(x) \approx 1$, so the proof is complete. $\square$

Remark. The above proof can be extended to show that $\lim_{b \to 1^-} S(x) = S(b)$, even if f is not continuous from the left at b. To do this, we just need to show that $S(x) \approx S(b)$ for *all* x satisfying $b \approx x < b$. The above proof showed that if M is a positive infinite hyperinteger, then $h^M(x) \approx 1$ implies $S(x) \approx S(b)$ for all x satisfying $t < x < b$. Therefore, it is enough to show that for all x satisfying $b \approx x < b$, we can find an infinite positive hyperinteger M such that $h^M(x) \approx 1$.

By the definition of derivative, if $0 \approx \varepsilon \neq 0$, then $\frac{\ln(1+\varepsilon)}{\varepsilon} \approx \ln'(1) = \frac{1}{1} = 1$. So, given any x satisfying $b \approx x < b$, we have $1 \approx h(x) < 1$, so $0 \approx h(x) - 1 < 0$, so we can set $\varepsilon = h(x) - 1$ to get $\frac{\ln(1+(h(x)-1))}{h(x)-1} \approx 1$, which implies $-1 \approx \frac{1}{1-h(x)} \ln(h(x))$, which in turn implies

$$\mathrm{e}^{-1} \approx \mathrm{e}^{(\ln(h(x)))(1/(1-h(x)))} = \left(\mathrm{e}^{\ln(h(x))}\right)^{1/(1-h(x))} = h(x)^{1/(1-h(x))}.$$

Since $1 \approx h(x) < 1$ implies $0 \approx 1 - h(x) > 0$, $1/\sqrt{1-h(x)}$ is positive infinite; let M be the greatest hyperinteger $\leq 1/\sqrt{1-h(x)}$. Therefore, since $0 < h(x) < 1$, we have

$$1 > h(x)^M \geq h(x)^{1/\sqrt{1-h(x)}} = h(x)^{\sqrt{1-h(x)}/(1-h(x))} = \left(h(x)^{1/(1-h(x))}\right)^{\sqrt{1-h(x)}} \approx (\mathrm{e}^{-1})^0 = 1,$$

so indeed $h^M(x) \approx 1$.