(1)

If a pair of six-sided dice are rolled 500 times, what is the approximate probability of a total of at least 3600? 3700? 3800?

$$P(x=1)=P(x=2)=\cdots=P(x=6)=\frac{1}{6}$$

$$P(y=1)=P(y=2)=\cdots=P(y=6)=\frac{1}{6}$$

$$\text{average}(y)=\text{average}(x)=(1+2+3+4+5+6)/6=3.5$$

$$\text{average}\left((x-3.5)^2\right)=\frac{2.5^2+1.5^2+.5^2+.5^2+1.5^2+2.5^2}{6}$$

$$\text{average}\left((y-3.5)^2\right)=\text{average}\left((x-3.5)^2\right)=\frac{35}{12}$$

$$\mu_x=\mu_y=\frac{7}{2}=3.5; \quad \sigma_x=\sigma_y=\sqrt{\frac{35}{12}}$$

$$Z_x=\frac{x-3.5}{\sqrt{35/12}} \qquad Z_y=\frac{y-3.5}{\sqrt{35/12}}$$

Given $b=3600$,

$$x_1+\cdots+x_{500}+y_1+\cdots+y_{500}\geq b$$

$$\iff (x_1 - 3.5) + \cdots + (y_{500} - 3.5) \ge b - 3500$$

$$\iff \frac{x_1 - 3.5}{\sqrt{35/12}} + \cdots + \frac{y_{500} - 3.5}{\sqrt{35/12}} \ge \frac{b - 3500}{\sqrt{35/12}}$$

$$\iff \frac{\left(\frac{x_1 - 3.5}{\sqrt{35/12}} + \cdots + \frac{x_{500} - 3.5}{\sqrt{35/12}} \right)}{\sqrt{500}} + \frac{\left(\frac{y_1 - 3.5}{\sqrt{35/12}} + \cdots + \frac{y_{500} - 3.5}{\sqrt{35/12}} \right)}{\sqrt{500}}$$

$$\ge \frac{1}{\sqrt{500}} \frac{b - 3500}{\sqrt{35/12}} = \sqrt{\frac{12}{35 \cdot 500}}\,(b - 3500)$$

Let $a = \sqrt{\frac{12}{35 \cdot 500}}\,(b - 3500)$.

By the Central Limit Theorem, the P.D.F. for

$$(u, v) := \left(\frac{Z_{x,1} + \cdots + Z_{x,500}}{\sqrt{500}}, \frac{Z_{y,1} + \cdots + Z_{y,500}}{\sqrt{500}} \right)$$

is $\approx \frac{e^{-u^2/2}}{\sqrt{2\pi}} \frac{e^{-v^2/2}}{\sqrt{2\pi}}$

$$= \frac{1}{2\pi} e^{-(u^2 + v^2)/2}$$

(3) $P(x_1 + \cdots + y_{500} \geq b) = P(u+v \geq a)$

$$\approx \iint_{u+v\geq a} e^{-(u^2+v^2)/2}\, dA / 2\pi$$

Let $\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$ $\therefore dA = r\,dr\,d\theta$

$$P(\ldots) \approx \iint_{r(\cos\theta+\sin\theta)\geq a} (1/2\pi)\, e^{-r^2/2}\, r\, dr\, d\theta$$

Since $0 \leq r < \infty$, always, $\infty \geq \frac{1}{r} > 0$

always, so $\cos\theta + \sin\theta \geq \frac{a}{r} > 0$

$\cos\theta + \sin\theta = 0 \iff \tan\theta = -1$

$\iff \theta = -\frac{\pi}{4}, -\frac{\pi}{4} \pm \pi, -\frac{\pi}{4} \pm 2\pi, \ldots$ Sign tests:

θ	$-\frac{5\pi}{4}$	$-\pi$	$-\frac{\pi}{4}$	0	$\frac{3\pi}{4}$	π	$\frac{7\pi}{4}$	2π	$\frac{11\pi}{4}$
$\cos\theta+\sin\theta$	0	-1	0	$+1$	0	-1	0	$+1$	0

To avoid double-counting, we restrict

θ to an interval of width $2\pi = 360°$

Any choice is okay, but it's easiest

to choose any interval such that the solution set for $\cos\theta + \sin\theta \geq 0$ is a single subinterval, not two subintervals. For example, restricting θ to $[-\pi, \pi]$, gives a solution set $\{\theta : -\pi/4 < \theta < 3\pi/4\} = \left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$.

$$P(\ldots) = \int_{-\pi/4}^{3\pi/4} \frac{1}{2\pi} \int_{a/(\cos\theta + \sin\theta)}^{\infty} r e^{-r^2/2} \, dr \, d\theta$$

because
$$\left\{\begin{array}{l} -\pi \leq \theta \leq \pi \\ 0 \leq r < \infty \\ r(\cos\theta + \sin\theta) \geq a \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} -\frac{\pi}{4} < \theta < \frac{3\pi}{4} \\ r \geq \frac{a}{\cos\theta + \sin\theta} \end{array}\right.$$

$u = -r^2/2 \Rightarrow du = -r\,dr$

$$P(\ldots) = \frac{1}{2\pi} \int_{-\pi/4}^{3\pi/4} \int_{-a^2/(\cos\theta + \sin\theta)^2}^{-\infty} e^{u} (-du) \, d\theta$$

$\lim_{u \to -\infty} e^{u} = 0$, so:

$$P(\ldots) = \frac{1}{2\pi} \int_{-\pi/4}^{3\pi/4} e^{-a^2/(\cos\theta + \sin\theta)^2} \, d\theta$$

(5) Numerical estimates of $\frac{1}{2\pi}\int_{-\pi/4}^{3\pi/4}(\ldots)\,d\theta$:

b	a	$P(x_1+\cdots+x_{500}+y_1+\cdots+y_{500}\geq b)\approx$
3600	2.61861	0.0044
3700	5.23723	8.2×10^{-8}
3800	7.85584	2.0×10^{-15}

Note: There is no known formula for the antiderivatives of $\exp(-a^2/(\cos\theta+\sin\theta)^2)$, so numerical estimates must be used.

Application: $P(x_1+\cdots+y_{500}\geq 3600) < \frac{1}{2}\%$, so if you get a total of ≥ 3600, you can be confident the dice are not fair or are not being rolled fairly.