# STOKES' THEOREM

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$$\begin{aligned}
\int_{\partial\Sigma}(P\,dx+Q\,dy+R\,dz) &= \iint_{\Sigma}\Big(dP\wedge dx+dQ\wedge dy+dR\wedge dz\Big)\\
&= \iint_{\Sigma}\left[\left(\frac{\partial P}{\partial x}dx+\frac{\partial P}{\partial y}dy+\frac{\partial P}{\partial z}dz\right)\wedge dx+\left(\frac{\partial Q}{\partial x}dx+\frac{\partial Q}{\partial y}dy+\frac{\partial Q}{\partial z}dz\right)\wedge dy+\left(\frac{\partial R}{\partial x}dx+\frac{\partial R}{\partial y}dy+\frac{\partial R}{\partial z}dz\right)\wedge dz\right]\\
&= \iint_{\Sigma}\left[\left(0-\frac{\partial P}{\partial y}dx\wedge dy+\frac{\partial P}{\partial z}dz\wedge dx\right)+\left(\frac{\partial Q}{\partial x}dx\wedge dy+0-\frac{\partial Q}{\partial z}dy\wedge dz\right)+\left(-\frac{\partial R}{\partial x}dz\wedge dx+\frac{\partial R}{\partial y}dy\wedge dz+0\right)\right]\\
&= \iint_{\Sigma}\left[\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\wedge dy+\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dy\wedge dz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dz\wedge dx\right]
\end{aligned}$$

**Abbrevations**

- One can use a single vector symbol, such as $\vec{F}$, to denote $\langle P,Q,R\rangle$.
- Likewise, $d\vec{r}$ is often used to represent $\langle dx,dy,dz\rangle$. If one wishes to emphasize the magnitude and direction of $d\vec{r}$, then one factors it as $\vec{T}\,ds$ where $\vec{T}$ is the unit tangent vector and $ds$ is the arc length differential $\sqrt{dx^2+dy^2+dz^2}$.
- The vector $\langle dy\wedge dz, dz\wedge dx, dx\wedge dy\rangle$ may be factored as $\vec{n}\,d\sigma$ where $\vec{n}$ the outward unit normal vector and $d\sigma$ is the surface area differential $\sqrt{(dy\wedge dz)^2+(dz\wedge dx)^2+(dx\wedge dy)^2}$.
- The vector $\left\langle \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}, \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right\rangle$ is often abbreviated as as $\vec{\nabla}\times\langle P,Q,R\rangle$ or curl $\langle P,Q,R\rangle$.
- Thus, Stokes' Theorem can be succincty written as $\int_{\partial\Sigma}\vec{F}\cdot\vec{T}\,ds=\iint_{\Sigma}\big(\vec{\nabla}\times\vec{F}\big)\cdot\vec{n}\,d\sigma$.
- To be even more succinct, write $\omega$ for $P\,dx+Q\,dy+R\,dz$ and $d\omega$ for $dP\wedge dx+dQ\wedge dy+dR\wedge dz$, producing $\int_{\partial\Sigma}\omega=\iint_{\Sigma}d\omega$.

**Assumptions**

- $\Sigma$ is assumed to be an oriented surface and $\partial\Sigma$ to be the positively oriented boundary of $\Sigma$.
- The unit outward normal vector $\vec{n}$ and is assumed to be continuous on $\Sigma$.
- The unit tangent vector $\vec{T}$ and is assumed to be continuous on $\partial\Sigma$.
- $\langle P,Q,R\rangle$ and its first partial derivatives are assumed to be continuous in an open region containing $\Sigma$ and $\partial\Sigma$.