Summary (days 10–19)

time $= t$; distance to origin $= |\vec{r}| = \sqrt{x^2+y^2+z^2}$ where

position $= \vec{r} = \langle x(t), y(t), z(t)\rangle$ (space curve)

velocity $= \vec{r}\,' = \langle x', y', z'\rangle = d\vec{r}/dt = \vec{v}$

acceleration $= \vec{r}\,'' = d\vec{v}/dt = \langle x'', y'', z''\rangle$

speed $= |\vec{v}| = |\vec{r}\,'| = |\vec{r}\,'\cdot\vec{r}\,'|^{1/2} = \sqrt{(x')^2+(y')^2+(z')^2}$

arc length $= \int_{t_{start}}^{t_{end}} \underbrace{(\text{speed})\,dt}_{ds} =$ (distance travelled for $t_{start} \le t \le t_{end}$)

$\vec{T}$ = unit tangent vector = direction of $\vec{v} = \hat{v} = \dfrac{\vec{r}\,'}{|\vec{r}\,'|}$

$\vec{N}$ = unit normal vector = direction of turning = direction of $\vec{T}\,' = \vec{T}\,'/|\vec{T}\,'|$

κ = curvature $= \left|\dfrac{d\vec{T}}{ds}\right| = \left|\dfrac{d\vec{T}/dt}{ds/dt}\right| = \dfrac{|\vec{T}\,'|}{|\vec{r}\,'|} = \dfrac{|\vec{r}\,'\times\vec{r}\,''|}{|\vec{r}\,'|^3}$

$R = 1/\kappa$ = radius of curvature. $\vec{T} \perp \vec{N}$.

At point $\vec{r}_0 = \vec{r}(t_0)$, where $\vec{r}$ is a space curve, if $\vec{T}_0 = \vec{T}(t_0)$ and $\vec{N}_0 = \vec{N}(t_0)$ and $R_0 = 1/\kappa$ at $t = t_0$, then the plane of osculation is given by position $\vec{r}_{plane} = \vec{r}_0 + h\vec{T}_0 + k\vec{N}_0$; $h, k \in \mathbb{R}$; the circle of osculation (kissing) is given by position $\vec{r}_{circle} = \vec{r}_0 + R_0(\vec{T}_0\cos\theta + \vec{N}_0(1+\sin\theta))$; $\theta \in [0, 2\pi]$.

$\vec{r}\cdot\vec{v} > 0 \iff |\vec{r}|$ increasing; $\vec{r}\cdot\vec{v} < 0 \iff |\vec{r}|$ decreasing

$\vec{v}\cdot\vec{a} > 0 \iff |\vec{v}|$ increasing; $\vec{v}\cdot\vec{a} < 0 \iff |\vec{v}|$ decreasing

Surface $z = f(x,y)$; "base point" (x_0, y_0).

Notation: $x = x_0 + \Delta x$, $y = y_0 + \Delta y$, $z_0 = f(x_0, y_0)$

$dx = \Delta x = x - x_0$, $\Delta y = y - y_0 = dy$

$z - z_0 = \Delta z = \Delta f = \Delta(f(x,y)) = f(x,y) - f(x_0, y_0)$

"f is continuous (cts.) at (x_0, y_0)" means there is a positive function δ such that

$$|\Delta f| < r \Leftarrow |\Delta x|, |\Delta y| \text{ (both)} < \delta(r)$$

"f is differentiable (diff.) at (x_0, y_0)" means there are functions $\varepsilon_1(\Delta x, \Delta y)$ & $\varepsilon_2(\Delta x, \Delta y)$ (called increments) and numbers $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ such that $\lim_{(\Delta x, \Delta y)\to(0,0)} \varepsilon_1 = 0 = \lim_{(\Delta x, \Delta y)\to(0,0)} \varepsilon_2$

and $$\Delta f = \left[\frac{\partial f}{\partial x}(x_0, y_0) + \varepsilon_1(\Delta x, \Delta y)\right]\Delta x + \left[\frac{\partial f}{\partial y}(x_0, y_0) + \varepsilon_2(\Delta x, \Delta y)\right]\Delta y.$$

Polynomials are differentiable (and continuous) everywhere. In particular, addition and multiplication are continuous:

$$|\Delta(x+y)| < r \Leftarrow |\Delta x|, |\Delta y| < \frac{r}{2}$$

$$|\Delta(xy)| < r \Leftarrow |\Delta x|, |\Delta y| < \min\{1, r/(|x_0| + |y_0| + 1)\}$$

$\partial f/\partial x$ = x partial derivative = df/dx with y constant.

$\partial f/\partial y$ = y partial derivative = df/dy with x constant.

If f is differentiable at (x_0, y_0), then:

Tangent plane of $z = f(x,y)$ at (x_0, y_0):

notation: $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

notation: $z_0 = f(x_0, y_0)$

$$z = z_0 + \boxed{dz} = \boxed{z_0 + \frac{\partial f}{\partial x}(x_0,y_0)\,dx + \frac{\partial f}{\partial y}(x_0,y_0)\,dy}$$

total differential

tangent plane approximation of z

or, equivalently, $\vec{r} - \vec{r}_0 \perp \left\langle \frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0), -1 \right\rangle$.

If $dx = 0$ or $dy = 0$, the tangent plane approximation is a tangent line approximation, which exists if the partial derivative exists.

A function could have partial derivatives at (x_0, y_0) but not be continuous there.

A function could be continuous and have partial derivatives at (x_0, y_0) but not be differentiable at (x_0, y_0).

If a function is differentiable at (x_0, y_0) then it is continuous and has partial derivatives at (x_0, y_0), and it has a tangent plane at (x_0, y_0).