Exact change $= \Delta \approx$ (total) differential $= d$ (for small $\Delta$)

% change $= \dfrac{\Delta z}{z} \approx \dfrac{dz}{z} = \dfrac{1}{z}\left(\dfrac{\partial z}{\partial x}\Delta x + \dfrac{\partial z}{\partial y}\Delta y\right)$

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Chain rule example: <u>if</u> $z = f(x,y)$, $x = g(t)$, and $h(t) = y$, <u>then</u> $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dg}{dt} + \dfrac{\partial f}{\partial y}\dfrac{h}{dt}$

Another: <u>if</u> $w = f(x, z)$, $z = g(x,y)$, <u>then</u>

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x} \quad \& \quad \frac{\partial w}{\partial y} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}$$

Another: <u>if</u> $w = f(z)$ & $z = g(x,y)$, <u>then</u>

$$\frac{\partial w}{\partial x} = \frac{df}{dz}\frac{\partial g}{\partial x} \quad \& \quad \frac{\partial w}{\partial y} = \frac{df}{dz}\frac{\partial g}{\partial y}.$$

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Higher partial derivatives: E.g. $f_{xyz} = \dfrac{\partial^3 f}{\partial z \partial y \partial x} =$

$z$-partial derivative of $y$-partial derivative of $x$-partial deriv. of $f$.

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$$g_{xy}(a,b) = \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} D \quad \& \quad g_{yx}(a,b) = \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} D$$

where $D = [g(a+\Delta x, b+\Delta y) - g(a+\Delta x, b) - g(a, b+\Delta y) + g(a,b)] / \Delta x \Delta y$

$g_{xy}(a,b) = g_{yx}(a,b)$ if $g_{xy}$ & $g_{yx}$ are continuous everywhere near $(a,b)$.

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The <u>direction</u> of a vector $\vec{v}$ is $\hat{v} = \vec{v}/|\vec{v}|$.

$D_{\hat{v}} f = \vec{\nabla} f \cdot \hat{v} =$ <u>directional derivative</u> (if $f$ is differentiable)

$\vec{\nabla} f = \left[\left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right\rangle \text{ (2D) or } \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}\right\rangle \text{ (3D)}\right] =$ <u>gradient</u>

$D_{\widehat{\vec{\nabla} f}} f$ maximizes $D_{\hat{v}} f$ at each point

$D_{-\widehat{\vec{\nabla} f}} f$ minimizes $D_{\hat{v}} f$ at each point.

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The line/plane/hyperplane tangent to $f(\vec{r}) = f(\vec{r}_0)$ at $\vec{r}_0$ is given by $(\vec{r} - \vec{r}_0) \cdot \vec{\nabla} f(\vec{r}_0) = 0$.

The line perpendicular/normal to $f(\vec{r}) = f(\vec{r}_0)$ at $\vec{r}_0$ is given by $\vec{r} = \vec{r}_0 + t \vec{\nabla} f(\vec{r}_0)$, $-\infty < t < \infty$.

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Local minima & maxima (extrema) of a differentiable $f(x,y)$ occur only where $\vec{\nabla} f = \vec{0} = \langle 0, 0 \rangle$. (And similarly for $f(x,y,z)$.) But $\vec{\nabla} f = \vec{0}$ does not guarantee a local extremum. Points where $\vec{\nabla} f = \vec{0}$ & $f_{xx} f_{yy} < f_{xy}^2$ are <u>saddle points</u> and there is no local extremum at these.

<u>If</u> $\vec{\nabla} f = \vec{0}$ & $f_{xx} f_{yy} > f_{xy}^2$, <u>then</u> $f_{xx} > 0$ guarantees we are at a local min. & $f_{xx} < 0$ guarantees we are at a local max. E.g., $xy$ has no local extremum; $x^2 + y^2 + 5$ has local min. 5 at $(0,0)$; $xy$ has saddle point $(0,0)$; $-x^2 - y^2 + 5$ has local max 5 at $(0,0)$.

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We can <u>sometimes</u> <u>approximately</u> locate a local min. by gradient descent: guess $\vec{r}_0$, then pick a step size $s$, then repeatedly improve: $\vec{r}_{n+1} = \vec{r}_n - s \vec{\nabla} f(\vec{r}_n)$.

(For finding local maxima, try $\vec{r}_{n+1} = \vec{r}_n + s \vec{\nabla} f(\vec{r}_n)$.)