Days 43–53 Summary

Spherical coordinates: (recall $r = \sqrt{x^2+y^2}$) distance to z-axis

$\rho = \sqrt{x^2+y^2+z^2} \in [0,\infty)$ (distance to origin)

$\phi = \arccos(z/\rho) \in [0,\pi]$ (colatitude)

$\theta = 2n\pi + \arccos\left(\frac{x}{r}\right) \cdot \operatorname{sign}(y)$ (longitude)

(Usually $\theta \in [0, 2\pi]$.) $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$x = \rho \sin\phi \cos\theta$ $\quad z = \rho\cos\phi$

$y = \rho \sin\phi \sin\theta$ $\quad r = \rho \sin\phi$

Vector Fields: $\vec{F} = \langle P(x,y), Q(x,y)\rangle$ (2D) or

$\vec{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ (3D)

$\text{curl}\, \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ (2D) $\quad \text{div}\, \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ (2D)

$d\vec{r} = \langle dx, dy\rangle$ (2D) $\quad \vec{n}\, ds = \langle -dy, dx\rangle$ (2D)

Green's Thm: $\int_{\partial D} \vec{F}\cdot d\vec{r} = \iint_D (\text{curl}\, \vec{F})\, dA$ and

$\int_{\partial D} \vec{F}\cdot \vec{n}\, ds = \iint_D (\text{div}\, \vec{F})\, dA$

$\text{curl}\, \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$ (3D)

$\text{div}\, \vec{F} = P_x + Q_y + R_z$ (3D) $\quad d\vec{r} = \langle dx, dy, dz\rangle$ (3D)

$\vec{\nabla} g(x,y) = \langle g_x, g_y\rangle$; $\vec{\nabla} g(x,y,z) = \langle g_x, g_y, g_z\rangle$

$\text{curl}\, \vec{\nabla} g = 0$ (2D); $\text{curl}\, \vec{\nabla} g = \vec{0}$ (3D)

2D anti-gradients: If D is a simply connected region (all in one piece & no holes) and $\text{curl}\,\vec{F} = 0$ on D, then $\vec{F} = \vec{\nabla} g$ where, for some fixed $(x_1, y_1) \in D$, for all $(x_2, y_2) \in D$ and all paths C through D from (x_1, y_1) to (x_2, y_2), $g(x_2, y_2) = \int_C \vec{F} \cdot d\vec{r}$.

3D anti-gradients: If $\text{curl}\,\vec{F} = \vec{0}$ on $\mathbb{R}^3$, then $\vec{F} = \vec{\nabla} g$ where $g(a,b,c) = \int_H \vec{F} \cdot d\vec{r}$ for every path H from $(0,0,0)$ to (a,b,c).

Fund. Thm. of Line Integrals: (2D and 3D)

$$\int_C \vec{\nabla} g \cdot d\vec{r} = g(\text{end}(C)) - g(\text{start}(C))$$

If C is a 2D path $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, then $\int_C (P(x,y)\,dx + Q(x,y)\,dy)$

$$= \int_{t=a}^{t=b} \Big(P(f(t), g(t))\, f'(t) + Q(f(t), g(t))\, g'(t)\Big)\, dt$$

and similarly for 3D paths (with $z = h(t)$, etc.)

The straight path from $\vec{a}$ to $\vec{b}$ can be parametrized by $\vec{r} = \vec{a}(1-t) + \vec{b}t$, $0 \leq t \leq 1$.

$\vec{r} = \langle x, y \rangle$ (2D); $\vec{r} = \langle x, y, z \rangle$ (3D).