HW51

$\langle P, Q\rangle$	P	Q	$P_x + Q_y$	$Q_x - P_y$
$\vec{F}$	$x+y$	y	2	-1
$\vec{G}$	y	x	0	0
$\vec{H}$	x^2	y^2	$2x+2y$	0

D in polar coordinates:

$0 \le r \le 5$

$0 \le \theta \le \pi$

$$\text{flux} = \int_{\partial D} \langle P, Q\rangle \cdot \underbrace{\vec{N}\, ds}_{\langle dy, -dx\rangle} = \iint_D (P_x + Q_y)\, dA$$

$$\text{circulation} = \int_{\partial D} \langle P, Q\rangle \cdot \underbrace{\vec{T}\, ds}_{\langle dx, dy\rangle} = \iint_D (Q_x - P_y)\, dA$$

Green's Thm.

Recall $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$.

$\langle P, Q\rangle$	$\vec{F}$	$\vec{G}$	$\vec{H}$
flux	$2 \cdot \text{area}(D) = 25\pi$	0	$\int_0^5 \left(\int_0^\pi 2r(\cos\theta + \sin\theta) r\, d\theta\right) dr = \frac{500}{3}$
circulation	$-1 \cdot \text{area}(D) = -\frac{25}{2}\pi$	0	0

HW52 $\operatorname{div}\langle P, Q, R\rangle = P_x + Q_y + R_z$

$\operatorname{curl}\langle P, Q, R\rangle = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$

P	Q	R	div	curl
y^2	z^2	x^2	0	$\langle -2z, -2x, -2y\rangle$
$x+2y$	$3y+4z$	$5z+6x$	9	$\langle -4, -6, -2\rangle$
x/ρ^4	y/ρ^4	z/ρ^4	$-1/\rho^4$	$\langle 0, 0, 0\rangle$

$\rho = \sqrt{x^2+y^2+z^2}$ after simplifying

The first field is incompressible ($\operatorname{div}=0$).
The third is irrotational ($\operatorname{curl}=\vec{0}$).

Note: irrotational $\Longleftrightarrow$ locally conservative.

HW54

① $A = \int_{-1}^{1}\left[\int_{-1}^{1} \sqrt{(-2x)^2+(-2y)^2+1^2}\, dy\right] dx \approx \boxed{7.446}$

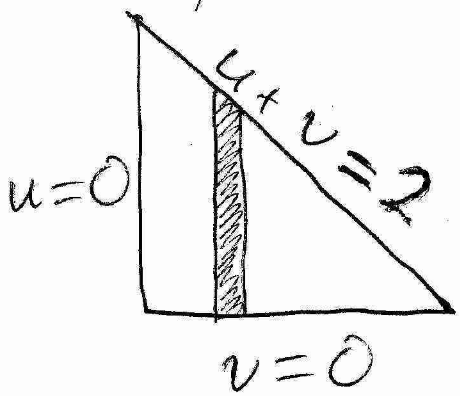
since $\vec{r}_x \times \vec{r}_y = \langle 1, 0, 2x \rangle \times \langle 0, 1, 2y \rangle = \langle -2x, -2y, 1 \rangle$

② $A = \int_0^{2\pi}\left[\int_0^{2\pi} (7+3\underbrace{\cos u}_{\left(\int_0^{2\pi}\cos u\, du = 0\right)})(3)\, d\theta\right] du = \boxed{(7)(3)(2\pi)^2} \approx 829$

since $\vec{r}_u \times \vec{r}_\theta = \langle -3\sin u \cos\theta, -3\sin u \sin\theta, 3\cos u \rangle$
$\times\ (7+3\cos u)\langle -\sin\theta, \cos\theta, 0 \rangle$
$= (7+3\cos u)(3)\underbrace{\langle -\cos u \cos\theta, -\cos u \sin\theta, -\sin u \rangle}_{\text{has magnitude } 1:}$

$\sqrt{1} = \sqrt{\cos^2 u + \sin^2 u} = \sqrt{\cos^2 u \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \sin^2 u}$

$$③ \quad A = \int_0^2 \left[\int_0^{2-u} \sqrt{\left[(6uv)^2 + (3(u+v))^2 + 2^2\right](u-v)^2} \, dv \right] du$$



since $\vec{r}_u \times \vec{r}_v = \langle 1, 2u, 3u^2 \rangle \times \langle -1, -2v, -3v^2 \rangle$

$= \langle 6u^2v - 6uv^2, 3v^2 - 3u^2, 2u - 2v \rangle$

$$A \approx \boxed{7.177}$$

① & ③ lack exact formulas, but the torus (donut) has enough symmetry for an exact formula. The torus also has an exact volume formula. (See Pappus' Theorem.)

HW55 Use spherical coordinates:

$\vec{r} = \langle 4\sin\varphi\cos\theta, 4\sin\varphi\sin\theta, 4\cos\varphi\rangle$

and $0 \le \varphi \le \pi/2$ & $0 \le \theta \le 2\pi$ parametrize S.

$$\vec{N}\,dA = \pm\,\vec{r}_\varphi \times \vec{r}_\theta \, d\varphi\, d\theta$$

$$= 4^2 \cdot \pm\langle \cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi\rangle$$

$$\times \langle -\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0\rangle\, d\varphi\, d\theta$$

$$= \pm 16\langle \sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\theta\rangle\, d\varphi\, d\theta$$

$$\Rightarrow \vec{N} = \pm\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\theta\rangle = \frac{\vec{N}\,dA}{|\vec{N}\,dA|}$$

$(|\vec{N}| = 1)$ (because, after simplifying, $|\vec{N}\,dA| = 16\sin\varphi\, d\varphi\, d\theta$)

At $\varphi = 0$, $\vec{r} = \langle 0,0,4\rangle$ and $\langle 0,0,1\rangle = N = \pm\langle 0,0,1\rangle$.

So, $\vec{N}\,dA = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle \cdot 16\sin\varphi\,d\varphi\,d\theta.$

$$\vec{F} = \langle z, 0, x^2\rangle = \langle 4\cos\varphi, 0, 16\sin^2\varphi\cos^2\theta\rangle.$$

$$\vec{F}\cdot\vec{N}\,dA = \left[4\cos\varphi\sin\varphi\cos\theta + 16\sin^2\varphi\cos\varphi\cos^2\theta\right] \cdot 16\sin\varphi\,d\varphi\,d\theta.$$

use $u = \sin\varphi$

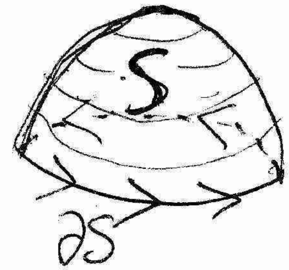
$$\int_0^{2\pi}\cos\theta\,d\theta = 0, \text{ so } \iint_S \vec{F}\cdot\vec{N}\,dA = \left[\int_0^{\frac{\pi}{2}} \underbrace{\sin^3\varphi}_{u^3}\,\underbrace{\cos\varphi\,d\varphi}_{du} \cdot \int_0^{2\pi}\cos^2\theta\,d\theta \cdot 256\right] = \int_0^1 u^3\,du \cdot \int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta \cdot 256$$

$$\left(0 = \int_0^{2\pi}\cos(2\theta)\,d\theta\right)$$

$$= \frac{1}{4}\cdot\pi\cdot 256$$

$$= \boxed{64\pi} \approx 201$$

HW56



$z=3$ & $r=1$ on ∂S

①

$$S: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 4 - r^2 \\ 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{cases} \qquad \partial S: \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = 3 \\ 0 \le \theta \le 2\pi \end{cases}$$

Note: $3 \le z \le 4 - r^2 \Rightarrow 3 \le 4 - r^2 \Leftrightarrow r^2 \le 1$

② $\vec{F} = \langle z - y, x, 2 \rangle.$ $\quad \vec{\nabla} \times \vec{F} = \langle 0, 1, 2 \rangle.$

$\vec{F} = \langle 3 - \sin\theta, \cos\theta, 2 \rangle$ on ∂S:

$\vec{T}\,ds = d\vec{r} = \langle dx, dy, dz \rangle = \langle -\sin\theta, \cos\theta, 0 \rangle\, d\theta$ on ∂S

$$\vec{N}\,dA = \pm \vec{r}_r \times \vec{r}_\theta \, dr\, d\theta$$
$$= \pm\, dr\, d\theta \langle \cos\theta, \sin\theta, -2r \rangle \times \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$\vec{N}\,dA = \pm r\,dr\,d\theta \langle 2r\cos\theta, 2r\sin\theta, 1\rangle$

Since $|\vec{N}| = 1$, we have $\vec{N} = \dfrac{\vec{N}\,dA}{|\vec{N}\,dA|}$

$= \pm\langle 2r\cos\theta, 2r\sin\theta, 1\rangle / \sqrt{4r^2+1} = \pm\langle 0,0,1\rangle$

at $r=0$. $\vec{N} = \langle 0,0,1\rangle$ at $(0,0,4) = (x,y,z)$

and $r=0 \Rightarrow (x,y,z) = (0,0,4)$, so,

$\vec{N}\,dA = r\,dr\,d\theta \langle 2r\cos\theta, 2r\sin\theta, 1\rangle$.

$(\vec{\nabla}\times\vec{F})\cdot\vec{N}\,dA = r\,dr\,d\theta[2r\sin\theta + 2]$

$\vec{F}\cdot\vec{T}\,ds = [3\sin\theta + 1]\,d\theta$

Note: $\left[\int_0^{2\pi} \sin\theta\,d\theta = 0\right]$

$\iint_S = \int_{r=0}^{r=1}\int_{\theta=0}^{\theta=2\pi}$; $\int_{\partial S} = \int_{\theta=0}^{\theta=2\pi}$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{N}\, dA = \int_0^1 \left(\int_0^{2\pi} 2r\, d\theta \right) dr$$

$$= \int_0^1 2r\, dr \int_0^{2\pi} d\theta = \boxed{2\pi}$$

$$\int_{\partial S} \vec{F} \cdot \vec{T}\, ds = \int_0^{2\pi} d\theta = \boxed{2\pi}$$

The above agreement is predicted by Stokes' Theorem:

$$\int_{\partial S} \vec{F} \cdot \vec{T}\, ds = \iint_S (\mathrm{curl}\, \vec{F}) \cdot \vec{N}\, dA$$