HW29

Let $f(x,y) = x^4 - 8x^2 + y^4 - 2y^2$.

Find and classify the nine critical points of f. Also find any local extrema.

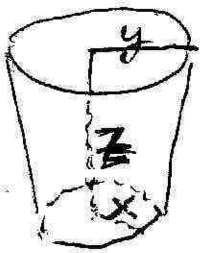
(The derivatives are simple enough that you can find the critical points by hand.)

HW30

① Find the point on the hyperbola $x^2 - y^2 + xy = 1$ closest to $(0, 1)$.

② Find the point on the ellipsoid $\frac{x^2}{1^2} + \frac{y^2}{2^2} + \frac{z^2}{3^2} = 1$ farthest from $(-3, 4, -5)$.

③ Reconsider the cup problem from HW27: Given a surface area of $S = 400\,\text{cm}^2$, what is the maximum possible volume V?



(Again: $S = \pi x^2 + \pi(x+y)\sqrt{z^2 + (y-x)^2}$;

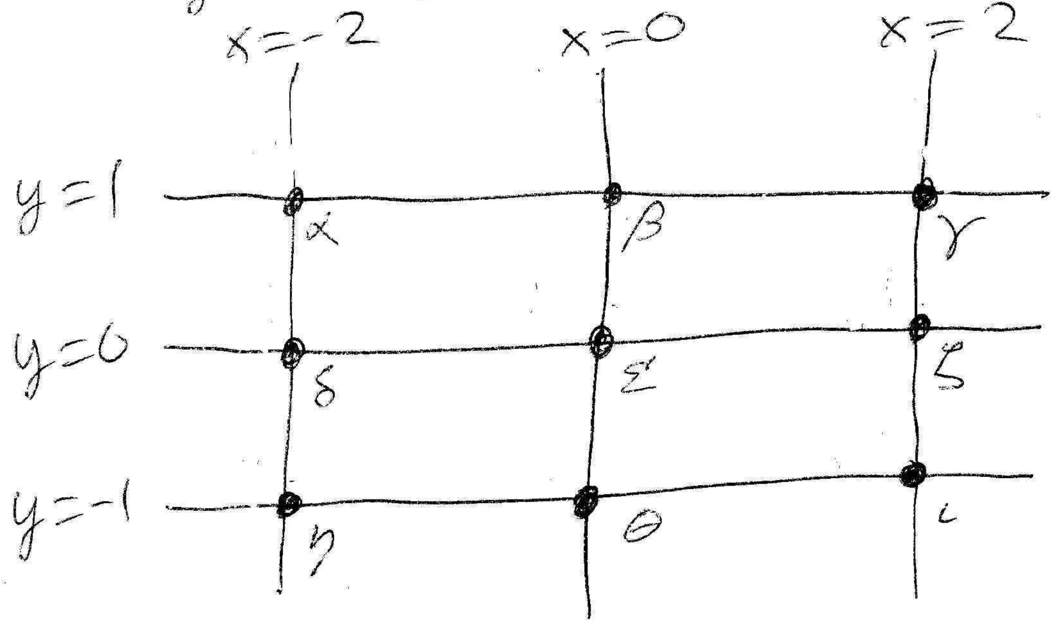
$V = \frac{\pi z}{3}\left(x^2 + xy + y^2\right)$.)

④ Find the min. & max. of $3x + 2y + z$ on the curve $\{(x, y, z) \mid x^2 + y^2 + z^5 = 5 \text{ and } xyz = 2\}$.

HW29

$$0 = f_x = 4x^3 - 16x = 4x(x-2)(x+2) \iff x = 0, \pm 2$$

$$0 = f_y = 4y^3 - 4y = 4y(y-1)(y+1) \iff y = 0, \pm 1$$



$$f_{xx} = 12x^2 - 16$$

$$f_{xy} = 0$$

$$f_{yy} = 12y^2 - 4$$

$$D = (12x^2-16)(12y^2-4)$$

$$= 16(3x^2-4)(3y^2-1)$$

point	α	β	γ	δ	ε	ζ	η	θ	ι
$3x^2-4$	+	−	+	+	−	+	+	−	+
$3y^2-1$	+	+	+	−	−	−	+	+	+
D	+	−	+	−	+	−	+	−	+
f_{xx}	+		+		−		+		+

HW29

Points $\beta, \delta, \zeta, \theta$ are saddle points.

At ε, f has local maximum $f(0,0) = 0$.

At $\alpha, \gamma, \eta, \iota$, f has local minimum $f(\pm 2, \pm 1) = -17$.

HW30

① Minimize $f(x,y) = x^2 + (y-1)^2$ subject to constraint $1 = g(x,y) = x^2 - y^2 + xy$.

Method: Find the "constrained critical points": $g = 1$ & $\vec{\nabla} f = \lambda \vec{\nabla} g$.

$\vec{\nabla} f = 2\langle x, y-1 \rangle$.

$\vec{\nabla} g = \langle 2x + y, x - 2y \rangle$.

Solve $\begin{cases} 1 = x^2 - y^2 + xy \\ 2x = \lambda(2x+y) \\ 2y - 2 = \lambda(x - 2y) \end{cases}$ Use a calculator or computer ...

Two solutions: $(x, y, \lambda) \approx \underbrace{(.937, .780, .706)}_{A}, \underbrace{(-1.12, .187, 1.09)}_{B}$

A has obviously the closer (x,y) to $(0,1)$. HW30

So, $(.937, .780)$ is $\approx$ the closest point requested.

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② Maximize $f(x,y,z)=(x+3)^2+(y-4)+(z+5)^2$

subject to constraint $1=g(x,y,z)=x^2+(y^2/4)+(z^2/9)$.

Again, we solve $g=1$ & $\vec{\nabla} f=\lambda \vec{\nabla} g$.

$$\vec{\nabla} f=2\langle x+3, y-4, z+5\rangle; \quad \vec{\nabla} g=2\left\langle x, \frac{y}{4}, \frac{z}{9}\right\rangle.$$

$$\text{Solve}\begin{cases} 1=x^2+\frac{1}{4}y^2+\frac{1}{9}z^2 \\ x+3=\lambda x \\ y-4=\lambda y/4 \\ z+5=\lambda z/9 \end{cases}$$

Again, a calculator or computer can find two solutions:

$$(x,y,z,\lambda)\approx(-.268, 1.13, -2.34, -10.2),$$
$$(.123, -.750, 2.76, 25.3)$$

$(.123, -7.50, 2.76)$ is clearly $\approx$ farthest point from $(-3,4,-5)$.

③ Maximize V subject to constraint $S=400$ by solving $\vec{\nabla}V=\lambda\vec{\nabla}S$ & $S=400$: HW30

$$\vec{\nabla}V=\frac{\pi}{3}\left\langle (2x+y)z,\ (x+2y)z,\ x^2+xy+y^2\right\rangle$$

$$\vec{\nabla}S=\langle 2\pi x,0,0\rangle+\pi\sqrt{z^2+(y-x)^2}\,\langle 1,1,0\rangle+\frac{\pi(x+y)\langle x-y,\ y-x,\ z\rangle}{\sqrt{z^2+(y-x)^2}}$$

(You can avoid having to even read the derivatives of S by storing them as variables.)

We solve $\begin{cases} S=400 \\ V_x=\lambda S_x \\ V_y=\lambda S_y \\ V_z=\lambda S_z \end{cases}$ using technology again:

$x\approx 4.55$ $\quad\lambda\approx .270$
$y\approx 8.30$
$z\approx 7.40$

At this (x,y,z), $V_{max}\approx\boxed{987\text{cm}^3}$. There is another solution: $(x,y,z,\lambda)\approx(0, 8.57, 12.13, .286)$, but there $V\approx 933$. Also $(x,y,z,\lambda)\approx(5.64, 0, 16.0, .501)$, but there $V\approx 532$.

HW30

(4) The objective (to be minimized/maximized) is $f(x,y,z) = 3x+2y+z$. The constraints are

$5 = g(x,y,z) = x^2+y^2+z^2$ & $2 = h(x,y,z) = xyz$.

We solve $\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h$ & $g=5$ & $h=2$.

$\vec{\nabla} f = \langle 3,2,1 \rangle$; $\vec{\nabla} g = 2\langle x,y,z \rangle$; $\vec{\nabla} h = \langle yz, zx, xy \rangle$.

Another system for our calculator or computer:

$$\begin{cases} 5 = x^2+y^2+z^2 \\ 2 = xyz \\ 3 = 2\lambda x + \mu yz \\ 2 = 2\lambda y + \mu zx \\ 1 = 2\lambda z + \mu xy \end{cases}$$

There are eight solutions: (We will not list λ & μ.)

x	y	z	$3x+2y+z$
-1.475	-1.344	1.009	-6.105
-1.085	-1.186	1.555	-4.072
-1.509	1.025	-1.294	-3.771
-1.057	1.541	-1.228	-1.316
1.006	-1.464	-1.358	-1.270
1.557	-1.094	-1.174	1.310
1.042	1.254	1.530	7.165
1.524	1.266	1.037	8.141

$\boxed{\text{minimum} \approx -6.105 \text{ \& maximum} \approx 8.141}$

HW31

① Find the volume of region $K = \{(x,y,z) \mid 0 \le x \le 1 \;\&\; 0 \le y \le 2 \;\&\; x^2 \le z \le 1 + y^2\}$.

② Compute $\int_0^2 \int_0^3 (x-y)^5 \, dx \, dy$ & $\int_0^2 \int_0^3 (x-y)^5 \, dy \, dx$.

③ Compute $\int_{-1}^2 \int_{-1}^3 xy^2 \, dx \, dy$ & $\int_{-1}^2 \int_{-1}^3 xy^2 \, dy \, dx$.

④ Compute $\int_0^1 \int_2^3 \ln(x^3 - y^3) \, dx \, dy$, approximately.

You'll need a calcutor or computer for ④.

HW3)

① $\int_0^1 \left(\int_0^2 (1+y^2-x^2)\,dy \right) dx = 4$

② $\int_0^2 \left(\int_0^3 (x-y)^5\,dx \right) dy = 49$

$\int_0^2 \left(\int_0^3 (x-y)^5\,dy \right) dx = -49$

③ $\int_{-1}^2 \left(\int_{-1}^3 xy^2\,dx \right) dy = 12$

$\int_{-1}^2 \left(\int_{-1}^3 xy^2\,dy \right) dx = 14$

④ $\int_0^1 \left(\int_2^3 \ln(x^3-y^3)\,dx \right) dy \approx 2.71086$

HW 32

① Find $\iint_T (x^2 + xy + y^2)\,dx\,dy$ where

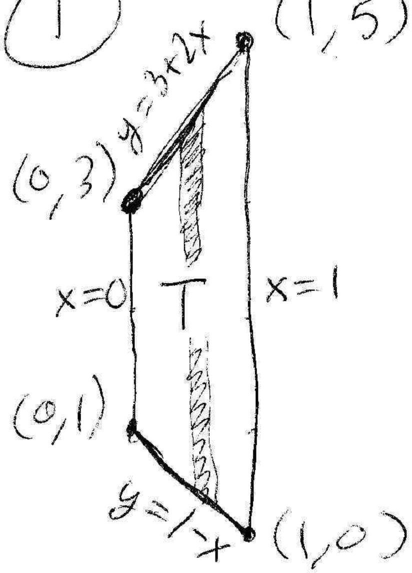
T is the trapezoid with vertices
$A = (0,1)$, $B = (1,0)$, $(1,5) = C$, $D = (0,3)$.

② Find the center of mass (x_{cm}, y_{cm}) of the region $\{(x,y) \mid 0 \leq x^3 \leq y \leq x^2\}$. (Sketch it to get a better idea; sketch the boundary curves first.)
$0 = x^3$, $x^3 = y$, and $y = x^2$

③ Find the average value of $\cos(x+2y)$ inside the triangle with vertices $(0,0)$, $(0, \frac{\pi}{4})$, $(\frac{\pi}{4}, 0)$.

HW 32

(1)

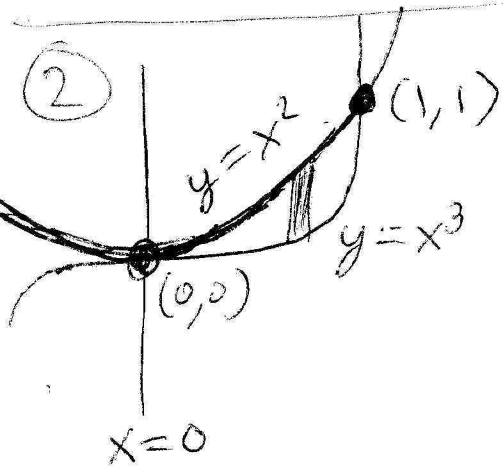


$$\iint_T = \int_{x=0}^{x=1} \int_{y=1-x}^{y=3+2x}$$

$$\iint_T (x^2+xy+y^2)\,dx\,dy = \int_0^1 \left(\int_{1-x}^{3+2x} (x^2+xy+y^2)\,dy \right) dx$$

$$= \boxed{\frac{689}{24} \approx 28.71}$$

(2)

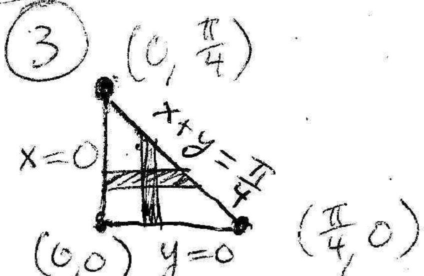


$$x_{cm} = \frac{\int_0^1 \left(\int_{x^3}^{x^2} x\,dy\right) dx}{\int_0^1 \left(\int_{x^3}^{x^2} 1\,dy\right) dx} = \frac{3}{5}$$

$$(x_{cm}, y_{cm}) = \left(\frac{3}{5}, \frac{12}{35}\right)$$

$$y_{cm} = \frac{\int_0^1 \left(\int_{x^3}^{x^2} y\,dy\right) dx}{\int_0^1 \left(\int_{x^3}^{x^2} 1\,dy\right) dx} = \frac{12}{35}$$

(3)



$$\iint_\Delta = \int_{x=0}^{x=\pi/4} \int_{y=0}^{y=(\pi/4)-x} = \int_{y=0}^{y=\pi/4} \int_{x=0}^{x=(\pi/4)-y}$$

Both work. Pick one.

$$\frac{\iint_\Delta \cos(x+2y)\,dx\,dy}{\iint_\Delta 1\,dx\,dy} = \boxed{\frac{16(\sqrt{2}-1)}{\pi^2} \approx 0.67}$$

HW33

(1) Find the volume of the region $\{(x,y,z) \mid 0 \le x^5 \le y^3 \le z \le 1\}$.

(2) Find the average value of x/y in the region $\{(x,y) \mid x+y \le 5 \ \& \ y-x^2 \ge 1 \ \& \ y \le 5\}$.

(3) Find $\iint_{\triangle ABC} e^{xy}\,dx\,dy$ where

$A=(0,1)$, $B=(1,0)$, $C=(2,2)$.

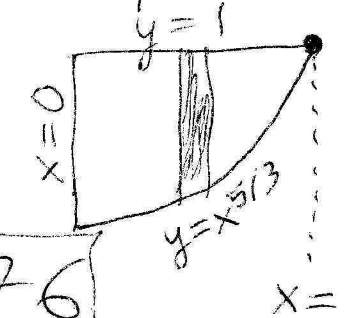
HW33

1. Given a tiny base $\underset{dx}{\square}\,dy$, the height of the column above it in our region is $1-y^3$.

So, $V=\iint_E (1-y^3)\,dx\,dy$ where $E=\{(x,y)\mid 0\le x^5 \le y^3 \le 1\}$.

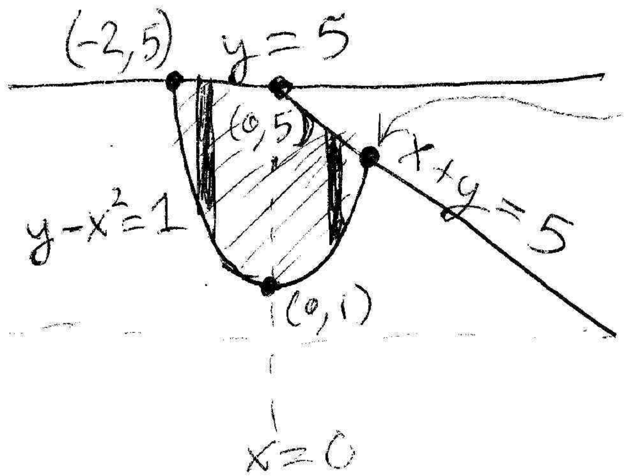
$0\le x \qquad x^{5/3}\le y\le 1$

For $x^{5/3}\le y\le 1$ to have a y-solution, we need $x^{5/3}\le 1$, which simplifies to $x\le 1$. So, $\iint_E = \int_{x=0}^{x=1}\int_{y=x^{5/3}}^{y=1}$.



So, $V=\int_0^1\left(\int_{x^{5/3}}^{1}(1-y^3)\,dy\right)dx = \boxed{\frac{75}{184}\approx .4076}$

2.



$x+y=5$ meets $y-x^2=1$ where $y=5-x$ & $\underbrace{5-x-x^2=1}$

$\frac{16}{4}=4=x^2+x=\left(x+\frac{1}{2}\right)^2-\frac{1}{4}$

$(x,y)=\left(\frac{-1}{2}\pm\frac{\sqrt{17}}{2},\ 5-\left(\frac{-1}{2}\pm\frac{\sqrt{17}}{2}\right)\right)$,

Choose $x>0$: $(x,y)=\left(\frac{\sqrt{17}-1}{2},\ \frac{11-\sqrt{17}}{2}\right)$

$$\iint = \iint + \iint = \int_{x=-2}^{x=0} \int_{y=1+x^2}^{y=5} + \int_{x=0}^{x=\frac{\sqrt{17}-1}{2}} \int_{y=1+x^2}^{y=5-x}$$

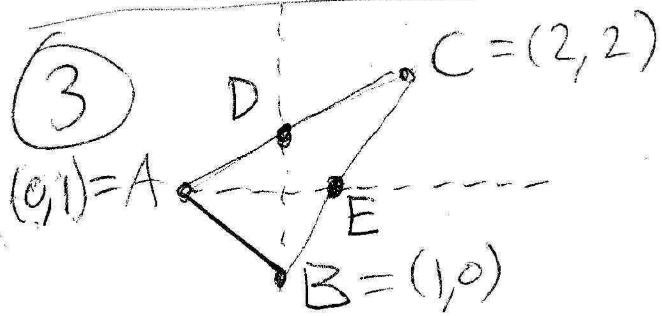
HW 33

In our region, the average value of x/y is

$$\left(\iint (x/y)\,dx\,dy\right) \Big/ \left(\iint 1\,dx\,dy\right) \approx \boxed{-0.04695}$$

(There is an exact formula, but you don't want to see it, trust me.)

3



AB line: $x = 1-y$ or $y = 1-x$

BC line: $x = 1+(y/2)$ or $y = 2x-2$

CA line: $x = 2y-2$ or $y = 1+(x/2)$

Choose horizontal or vertical slicing.

Horizontal: $\iint_{ABC} = \iint_{ABE} + \iint_{AEC} = \int_{y=0}^{y=1} \int_{x=1-y}^{x=1+y/2} + \int_{y=1}^{y=2} \int_{x=2y-2}^{x=1+y/2}$

Vertical: $\iint_{ABC} = \iint_{ABD} + \iint_{BCD} = \int_{x=0}^{x=1} \int_{y=1-x}^{y=1+\frac{x}{2}} + \int_{x=1}^{x=2} \int_{y=2x-2}^{y=1+x/2}$

Either way, $\iint_{ABC} e^{xy}\,dx\,dy \approx \boxed{6.738}$