HW 34

(1) Find the center of mass of the region $\{(x,y) \mid x, y \geq 0 \ \& \ 4 \leq x^2 + y^2 \leq 9\}$.

(2) Check out this trick: An important integral from statistics, $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx$, equals $\sqrt{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy}$

$$= \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy} = \sqrt{\int_0^{\infty}\int_0^{2\pi} e^{-r^2/2}\, r\,d\theta\,dr}$$

$$= \sqrt{2\pi \int_0^{-\infty} e^{u}(-du)} = \sqrt{2\pi(-(0-1))} = \sqrt{2\pi}$$

Use a similar trick to evaluate

$\int_0^{\infty}\int_0^{\infty} \frac{dx\,dy}{(x^2+y^2)^2+1}$. You don't need the $\sqrt{\ }$ part of the trick. But polar coordinates will really help!

(3) Find $\iint_S xy\,dx\,dy$ where $S = \{(x,y) \mid r \leq \theta \leq \pi\}$.

HW34

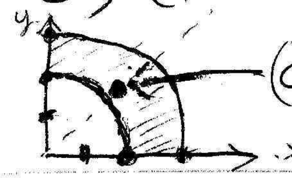
① $4 \le x^2+y^2 \le 9 \iff 2 \le r \le 3$

$x, y \ge 0 \iff 0 \le \theta \le \pi/2$

$$x_{cm} = \frac{\int_0^{\pi/2}\int_2^3 \overbrace{r\cos\theta}^{x}\,\overbrace{r\,dr\,d\theta}^{dA}}{\int_0^{\pi/2}\int_2^3 \underbrace{r\,dr\,d\theta}_{dA}} = \frac{\int_0^{\pi/2}\cos\theta\,d\theta\int_2^3 r^2\,dr}{\int_0^{\pi/2}d\theta\int_2^3 r\,dr}$$

$$= \frac{\left(\sin\theta\big|_0^{\pi/2}\right)\left((r^3/3)\big|_2^3\right)}{\left(\theta\big|_0^{\pi/2}\right)\left((r^2/2)\big|_2^3\right)} = \frac{(1-0)(27-8)/3}{(\frac{\pi}{2}-0)(9-4)/2} = \boxed{\frac{76}{15\pi}}$$

By symmetry, $\boxed{y_{cm} = x_{cm}}$.



(c.m. has $\underbrace{\theta = \pi/4}_{\hookrightarrow\, y=x}$)

② $$\int_0^\infty\int_0^\infty \frac{dx\,dy}{r^4+1} = \int_0^{\pi/2}\int_0^\infty \frac{r\,dr\,d\theta}{r^4+1} = \underbrace{\int_0^{\pi/2} d\theta}_{\pi/2}\int_0^\infty \frac{r\,dr}{r^4+1}$$

$$= \left[\frac{\pi}{4}\int_0^\infty \frac{du}{u^2+1} \text{ (using } u=r^2)\right] = \left[\frac{\pi}{4}\int_0^{\pi/2} d\omega \text{ (using } u=\tan\omega)\right]$$

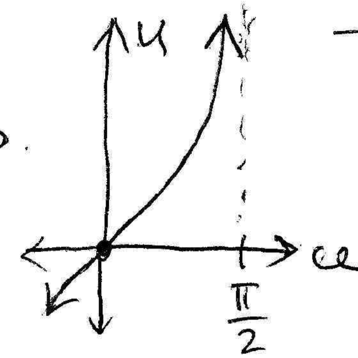
$$= \boxed{\frac{\pi^2}{8}}$$

$du = 2r\,dr$

$du/2 = r\,dr$

$du = \sec^2\omega\,d\omega$

$u^2+1 = \sec^2\omega$



HW34

③ $\iint_S = \int_{\theta=0}^{\theta=\pi}\boxed{2}\int_{r=0}^{r=\theta}\boxed{1}$ because: $\boxed{1}$ Given θ, $\rightarrow$ $0\le r\le\theta$.

(Note: $r=\sqrt{x^2+y^2}\ge 0$ always.) $\boxed{2}$ For $0\le r\le\theta\le\pi$ to have a solution at all, we need $0\le\theta\le\pi$.

$$\iint_S xy \underbrace{dx\,dy}_{dA_{\text{rectangular}}} = \int_0^\pi\int_0^\theta \underbrace{r^2\cos\theta\sin\theta}_{xy}\,\underbrace{r\,dr\,d\theta}_{dA_{\text{polar}}}$$

$$=\int_0^\pi\left[\left(\frac{1}{2}\sin(2\theta)\right)\int_0^\theta r^3\,dr\right]d\theta=\int_0^\pi\frac{\theta^4}{8}\sin(2\theta)\,d\theta$$

Integrate by parts 4 times:

$\theta^4/8$	$\sin(2\theta)$	
$\theta^3/2$	$-\frac{1}{2}\cos 2\theta$	$+$
$3\theta^2/2$	$-\frac{1}{4}\sin 2\theta$	$-$
3θ	$\frac{1}{8}\cos 2\theta$	$+$
3	$\frac{1}{16}\sin 2\theta$	$-$
0	$-\frac{1}{32}\cos 2\theta$	$+$

$$\left[-\frac{\theta^4}{16}\cos 2\theta+\frac{\theta^3}{8}\sin 3\theta+\frac{3\theta^2}{16}\cos 2\theta-\frac{3\theta}{16}\sin 2\theta-\frac{3}{32}\cos 2\theta\right]\Bigg|_0^\pi$$

$$\boxed{=-\frac{\pi^4}{16}+\frac{3\pi^2}{16}}$$

① Find $\int_1^2 \int_3^4 \int_5^6 \ln(x+2y+3z)\,dx\,dz\,dy$. HW35

② The region of integration for ① is $[\_\_, \_\_] \times [\_\_, \_\_] \times [\_\_, \_\_]$.

(Fill in the blanks.)

③ Find average value of $\varphi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$ in the cube $[1, 5]^3$. You may need a calculator/computer to find a decimal answer. I don't believe there is an exact formula.

HW35

① $\int_1^2\left(\int_3^4\left(\int_5^6 \ln(x+2y+3z)\,dx\right)dz\right)dy \approx 2.9428$

(It's lots of tedious integration by parts to get the very complicated exact formula — not worth it.)

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② $[5,6]\times[1,2]\times[3,4]$

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③ $\iiint_{[1,5]^3} \varphi\, dV = \int_1^5\left(\int_1^5\left(\int_1^5 \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)dz\right)dy\right)dx$

≈ 62.2215

$\iiint_{[1,5]^3} dV = \int_1^5\left(\underbrace{\int_1^5\left(\overbrace{\int_1^5 1\,dz}^{4}\right)dy}_{16}\right)dx = 64.$

(Or use geometry: volume $=(5-1)^3$.)

$\text{average}(\varphi) \approx \frac{62.2215}{64} \approx 0.972211$

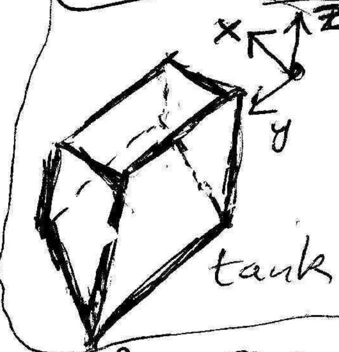
HW36

① If a tank of fluid has density $dm/dV = 5-2z$ (so, $dm = (5-2z)dV$) and its bounds are given by $0 \le x \le 10$, $0 \le y \le 20$, and $3x - 80 \le z \le 0$, then what is the total mass $\iiint_{tank} dm$?

② Find the average value of $\rho^2 = x^2+y^2+z^2$ in the region $B = \{(x,y,z) \mid 1 \le x, y, z$ & $x+y+z \le 12\}$.

③ Find $\iiint_E ze^x \, dV$ where $E = \{(x,y,z) \mid x^2 \le 1 \le y \le 3z \le 6\}$

HW36



① $$\iiint_{tank} = \int_{x=0}^{x=10^{*}} \int_{y=0}^{y=20} \int_{z=3x-80}^{z=0}$$

$$\iiint_{tank} dm = \underbrace{\int_0^{20} dy}_{20} \cdot \int_0^{10} \Big(\underbrace{\int_{3x-80}^{0} (5-2z)\, dz}_{-(5z-z^2)\big|_0^{3x-80}} \Big) dx$$

$$\begin{cases} u = 3x-80 \\ du = 3\,dx \\ du/3 = dx \end{cases}$$

$$= 20 \int_0^{10} \left[(3x-80)^2 - 5(3x-80) \right] dx$$

$$= 20 \int_{-80}^{-50} \left[u^2 - 5u \right] du/3$$

$$= 20 \left(u^3/9 - 5u^2/6 \right) \Big|_{-80}^{-50} = \left[\frac{u^2}{3} \left(\frac{u}{3} - \frac{5}{2} \right) \right] \Big|_{-80}^{-50} \cdot 20$$

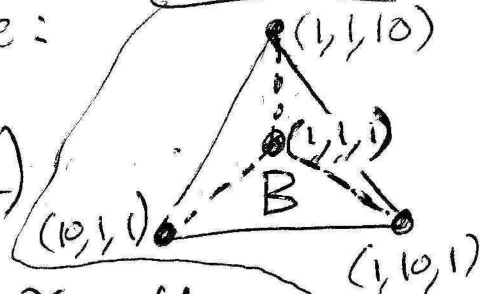
$$= 20 \left(\frac{50^2}{3} \left(-\frac{50}{3} - \frac{5}{2} \right) - \frac{80^2}{3} \left(-\frac{80}{3} - \frac{5}{2} \right) \right) = \boxed{925{,}000}$$

$*$ $0 \le x \le 10 \Rightarrow -80 \le 3x-80 \le -50$

$\Rightarrow 3x-80 \le z \le 0$ has a solution

HW36

(2) $\iiint_B = \int_{x=1}^{x=10} \boxed{3} \int_{y=1}^{y=11-x} \boxed{2} \int_{z=1}^{12-x-y} \boxed{1}$ because:



$\boxed{1}$ Given x & y, $(1 \le z \text{ \& } x+y+z \le 12)$

$\iff 1 \le z \le 12-x-y$

$\boxed{2}$ Given x, $1 \le y$ & for $1 \le z \le 12-x-y$

to have a z-solution, we need $1 \le 12-x-y$,

which rearranges: $y \le 12-x-1 = 11-x$

$\boxed{3}$ For $1 \le y \le 11-x$ to have a y-solution,

we need $1 \le 11-x$, which rearranges: $x \le \overbrace{11-1}^{10}$.

(And $1 \le x$ is required by B.)

$$\iiint_B \rho^2 \, dV \Big/ \iiint_B 1 \, dV = \frac{\int_1^{10}\left(\int_1^{11-x}\left(\int_1^{12-x-y}(x^2+y^2+z^2)\,dz\right)dy\right)dx}{\underbrace{9^3/6}}$$

← volume(B) from geometry or doing the integral.

$$\boxed{= \frac{204}{5} = 40.8}$$

③ $\iiint_E = \int_{x=-1}^{x=1} \boxed{3} \int_{z=1/3}^{z=2} \boxed{2} \int_{y=1}^{y=3z} \boxed{1}$ because:

$\boxed{\text{HW36}}$

$\boxed{1}$ Given x & z, E requires $1 \le y \le 3z$.

$\boxed{2}$ Given x, for $1 \le y \le 3z \le 6$ to have any y-solution, we need $1 \le 3z \le 6 \left(\Leftrightarrow \frac{1}{3} \le z \le 2 \right)$.

$\boxed{3}$ For $\begin{cases} x^2 \le 1 \\ 1 \le 3z \le 6 \end{cases}$ to have a solution, we need $-1 \le x \le 1$.
Note: $1 \le 3z \le 6$ doesn't depend on x.

$$\iiint_E z e^x \, dV = \underbrace{\int_{-1}^{1} e^x \, dx}_{e^1 - e^{-1}} \cdot \int_{1/3}^{2} \underbrace{\left(z \cdot \underbrace{\int_1^{3z} dy}_{3z-1} \right)}_{3z^2 - z} dz$$

$$= \left(e - \frac{1}{e}\right)\left(z^3 - \frac{z^2}{2}\right)\Big|_{1/3}^{2} = \boxed{\frac{325(e^2-1)}{54e} \approx 14.1459}$$

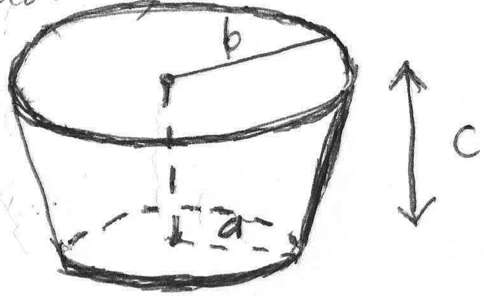
① Let $E = \{(x,y,z) \mid 0 \le z \le x+y \ \& \ r \le 4\}$. **HW37**

Find the average x-coordinate in E.

Hint for ①: Where $x+y=0$, what values in $[-\pi,\pi]$ does θ take?

② Find $\iiint_H xy\, dV$ where $H = \{(x,y,z) \mid r \le z \le \theta \le \pi\}$.

③ Recall the cup from previous homeworks, a truncated cone. For less confusion with the coordinate axes, let a be the smaller radius, b the larger radius, and c the height:



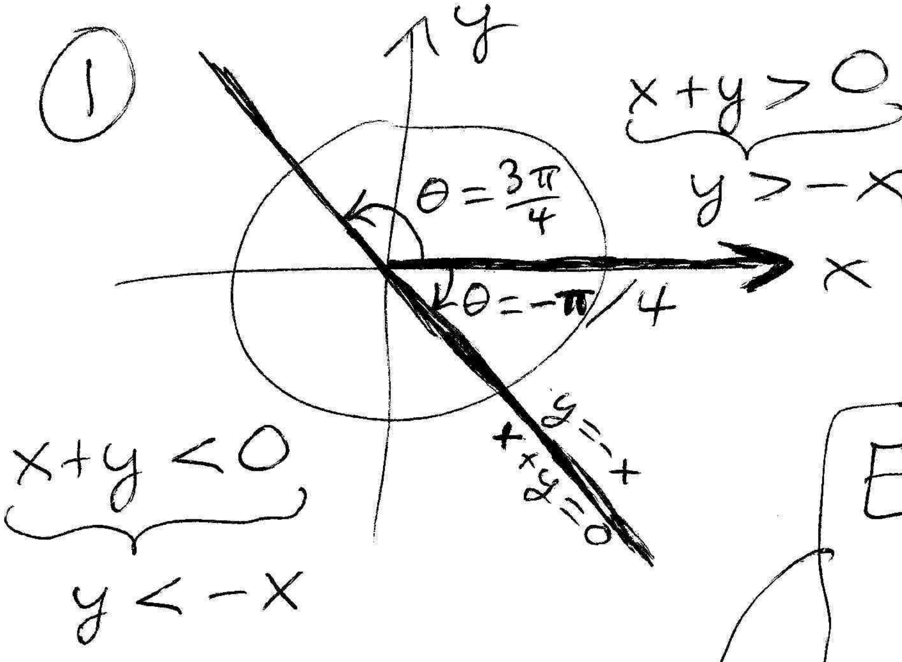
Find an integral of the form

$$\int_{\theta=0}^{\theta=2\pi}\left(\int_{z=\square}^{z=\square}\left(\int_{r=\square}^{r=\square} \square \, dr\right)dz\right)d\theta$$

equal to the volume of the cup.

HW 37

1



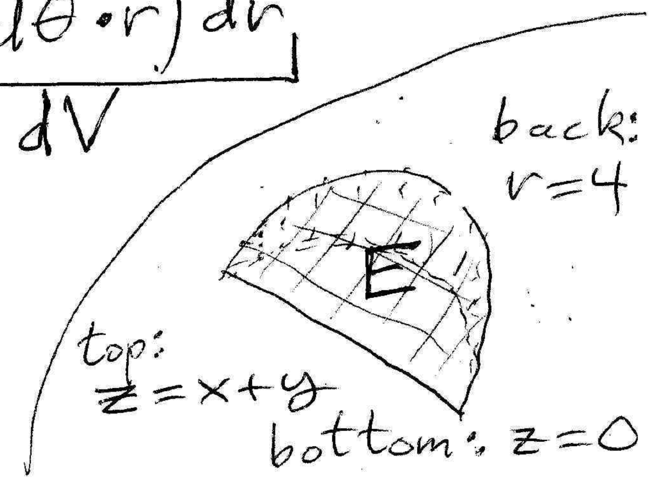
$$\boxed{x+y\geq 0 \iff -\frac{\pi}{4}\leq\theta\leq\frac{3\pi}{4}}$$

$$\boxed{E=\Big\{(x,y,z)\,\Big|\,0\leq z\leq \underbrace{r(\cos\theta+\sin\theta)}_{x+y} \quad 0\leq\sqrt{x^2+y^2}=r\leq 4 \;\&\; -\frac{\pi}{4}\leq\theta\leq\frac{3\pi}{4}\Big\}}$$

This is because for $0\leq z\leq x+y$ to have a z-solution, we need $0\leq x+y$.

$$\frac{\iiint_E x\,dV}{\iiint_E 1\,dV}=\frac{\int_0^4\left(\int_{-\pi/4}^{3\pi/4}\left(\int_0^{r\cos\theta+r\sin\theta}\overbrace{r\cos\theta}^{x}\,dz\right)d\theta\cdot r\right)dr}{\int_0^4\left(\int_{-\pi/4}^{3\pi/4}\left(\int_0^{r\cos\theta+r\sin\theta}\underbrace{dz\Big)d\theta\cdot r\Big)dr}_{dV}\right.\right.}$$

$$\boxed{=\frac{32\pi}{128\sqrt{2}/3}=\frac{3\pi}{4\sqrt{2}}\approx 1.66608}$$



HW 37

(2) $\iiint_H = \int_{\theta=0}^{\theta=\pi} \boxed{1} \int_{z=0}^{z=\theta} \boxed{2} \int_{r=0}^{r=z} \boxed{3}$ because:

$\boxed{1}$ Given θ & z, H requires $r \leq z$. $r = \sqrt{x^2+y^2} \geq 0$ always.

$\boxed{2}$ Given θ, for $0 \leq r \leq z$ to have an r-solution requires $0 \leq z$. H also requires $z \leq \theta$.

$\boxed{3}$ For $0 \leq z \leq \theta$ to have a z-solution requires $0 \leq \theta$. H also requires $\theta \leq \pi$.

$$\iiint_H xy\, dV = \int_0^{\pi}\left(\int_0^{\theta}\left(\int_0^{z} (r\cos\theta)(r\sin\theta)\, r\, dr\right) dz\right) d\theta$$

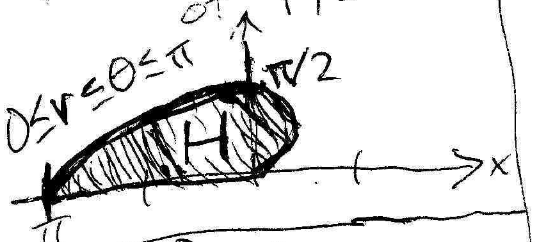
$$= \int_0^{\pi}\left(\tfrac{1}{2}\sin 2\theta \cdot \underbrace{\int_0^{\theta}\underbrace{\left(\int_0^{z} r^3\, dr\right)}_{z^4/4} dz}_{\theta^5/20}\right) d\theta$$

$$= \frac{1}{40}\int_0^{\pi} \theta^5 \sin 2\theta\, d\theta = \boxed{-\frac{\pi^5}{80} - \frac{3\pi}{32} + \frac{\pi^3}{16}}$$

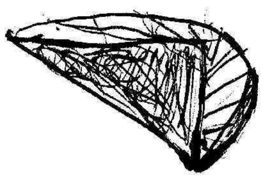
(Integrate by parts five times...)

$$\approx \boxed{-2.182}$$

Overhead view of H:

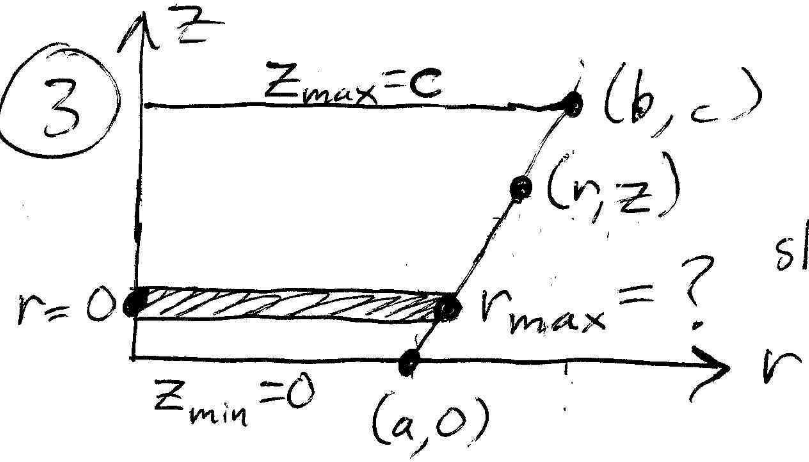


3D view of H: (It's weird.)



HW37

3



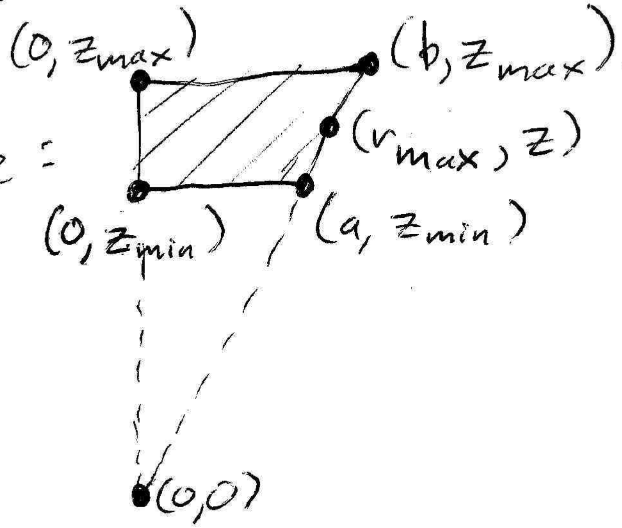
On slanted line:

$$\text{slope}=\frac{z-0}{r-a}=\frac{c-0}{b-a}\Rightarrow r=a+\left(\frac{b-a}{c}\right)z$$

$$\text{volume(cup)}=\int_{\theta=0}^{\theta=2\pi}\int_{z=\boxed{0}}^{z=\boxed{c}}\int_{r=\boxed{0}}^{r=\boxed{a+\left(\frac{b-a}{c}\right)z}}\underbrace{\boxed{r}\,dr\,dz\,d\theta}_{dV}$$

There are other solutions because the plane $z=0$ doesn't have to be at the bottom of the cup.

For example:

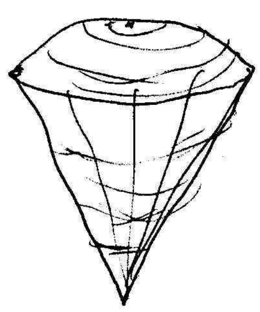


← In this alternative coordinate system,

$$\begin{cases} r_{max}=\left(\frac{b-a}{c}\right)z \\ z_{max}=\frac{bc}{b-a} \\ z_{min}=\frac{ac}{b-a} \end{cases}$$

HW38

1. Find the average z-coordinate inside the following cone with a spherical cap:



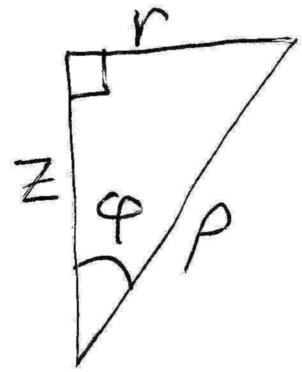
$$C = \{(x,y,z) \mid \sqrt{2(x^2+y^2)} \le z \ \& \ x^2+y^2+z^2 \le 1\}.$$

2. Find the average y-coordinate inside the hemisphere $H = \{(x,y,z) \mid y \ge 0 \ \& \ x^2+y^2+z^2 \le 16\}$

3. Simple models of Earth's atmosphere describe the density of air as $\frac{dm}{dV} = Ae^{-B\rho}$ for $\rho \ge R$ where A & B are positive constants & $\rho = \sqrt{x^2+y^2+z^2}$ is distance to the Earth's center at $(0,0,0)$ & R is Earth's radius. Find a formula $M(A,B,R)$ for the mass $\iiint_K dm$ of the atmosphere where

$$K = \{(x,y,z) \mid \rho \ge R\}.$$

(1)



HW 38

$\underline{\text{If}}$ $z>0$, then $\varphi = \tan^{-1}\left(\frac{r}{z}\right)$

(If $z<0$, then $\varphi = \pi + \tan^{-1}\left(\frac{r}{z}\right)$.)

$$\sqrt{2(x^2+y^2)} \le z \iff \sqrt{2}\, r \le z \iff \frac{r}{z} \le \frac{1}{\sqrt{2}}$$

$$\iff \varphi \le \underbrace{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)}_{\text{Call this } \beta.}.$$

Therefore, $\iiint_C = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\beta} \int_{\rho=0}^{\rho=1} \cdot$

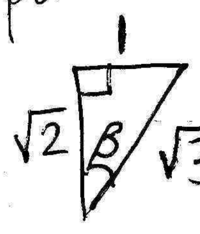
$$\left[z\, dV = (\rho\cos\varphi)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta\right]$$

$$\frac{\iiint_C z\, dV}{\iiint_C dV} = \frac{\cancel{\int_0^{2\pi} d\theta} \int_0^1 \rho^3\, d\rho \int_0^\beta \sin\varphi\cos\varphi\, d\varphi}{\cancel{\int_0^{2\pi} d\theta} \int_0^1 \rho^2\, d\rho \int_0^\beta \sin\varphi\, d\varphi} \leftarrow \left[dV = \rho^2 \sin\varphi \cdot d\rho\, d\varphi\, d\theta\right]$$

$$= \frac{(1/4)\left(\frac{1}{4}\sin 2\beta\right)}{(1/3)(1-\cos\beta)} = \boxed{\frac{3\sin\left(2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)}{16\left(1-\cos\left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)\right)}} = \frac{\sqrt{6}}{8(\sqrt{3}-\sqrt{2})}$$

$$\boxed{\approx 0.9633}$$

Optional simplification:



$\sqrt{3} = \sqrt{\sqrt{2}^2 + 1^2}$

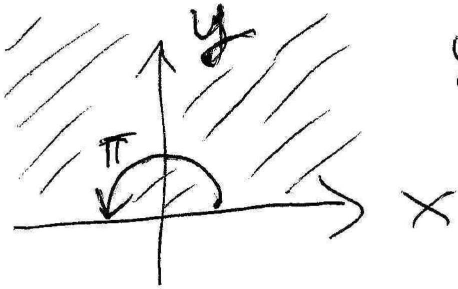
$\cos\beta = \sqrt{2/3}$

$\sin\beta = 1/\sqrt{3}$

$$\Longrightarrow \sin 2\beta = 2\cos\beta\sin\beta = \frac{2\sqrt{2}}{3}$$

HW 38

(2) $y \geq 0 \iff 0 \leq \theta \leq \pi$



$$\iiint_H = \int_{\theta=0}^{\theta=\pi} \underbrace{\int_{\varphi=0}^{\varphi=\pi} \int_{\rho=0}^{\rho=4}}$$

$\rho \geq 0$ always: $\rho = \sqrt{x^2+y^2+z^2}$

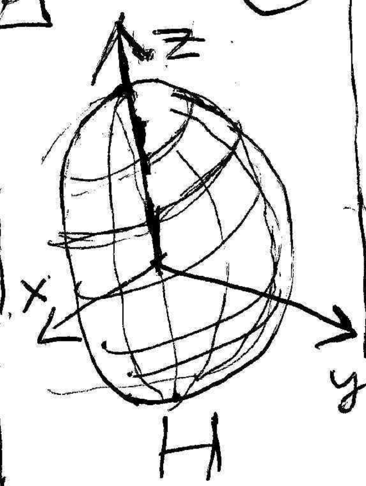
&

$0 \leq \varphi \leq \pi$ always: $\varphi = \cos^{-1}\left(\frac{z}{\rho}\right)$

$(y = r\sin\theta = \rho\sin\varphi\sin\theta)$

$$\frac{\iiint_H y\,dV}{\iiint_H dV} = \frac{\overbrace{\int_0^\pi \sin\theta\,d\theta}^{2}\ \overbrace{\int_0^\pi \sin^2\varphi\,d\varphi}^{\pi/2}\ \overbrace{\int_0^4 \rho^3\,d\rho}^{64}}{\underbrace{\int_0^\pi d\theta}_{\pi}\ \underbrace{\int_0^\pi \sin\varphi\,d\varphi}_{2}\ \underbrace{\int_0^4 \rho^2\,d\rho}_{64/3}} = \frac{3}{2}$$

$(dV = \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta)$



(3) $$\iiint_K dm = \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\,d\varphi \int_R^{\infty} A\rho^2 e^{(-B\rho)}\,d\rho$$

$$dm = \underbrace{\left(Ae^{-B\rho}\right)}_{dm/dV}\underbrace{\left(\rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta\right)}_{dV}$$

$$\Rightarrow \iiint_K dm = (2\pi)(2)\left[A\left(-\frac{\rho^2}{B} - \frac{2\rho}{B^2} - \frac{2}{B^3}\right)e^{-B\rho}\right]\Bigg|_{\rho=R}^{\rho\to\infty}$$

$$\Rightarrow M(A,B,R) = \boxed{4\pi A e^{-BR}\left((BR)^2 + 2BR + 2\right)/B^3}$$